hep-th/0003298
PAR-LPTHE 00-09
ETH-TH/00-2
CERN-TH/2000-0045
March 2000

PROJECTIONS IN STRING THEORY AND BOUNDARY STATES FOR GEPNER MODELS

Jürgen Fuchs [1], Christoph Schweigert [2], Johannes Walcher [3,4]

[1] Institutionen för fysik
Universitetsgatan 1
S – 651 88 Karlstad

[2] LPTHE, Université Paris VI
4 place Jussieu
F – 75 252 Paris Cedex 05

[3] Institut für Theoretische Physik
ETH Hönggerberg
CH – 8093 Zürich

[4] Theory Division, CERN
CH – 1211 Genève 23

Abstract

In string theory various projections have to be imposed to ensure supersymmetry. We study the consequences of these projections in the presence of world sheet boundaries. A-type boundary conditions come in several classes; only boundary fields that do not change the class preserve supersymmetry. Our analysis takes in particular properly into account the resolution of fixed points under the projections. Thus e.g. the compositeness of some previously considered boundary states of Gepner models follows from chiral properties of the projections. Our arguments are model independent; in particular, integrality of all annulus coefficients is ensured by model independent arguments.

1 Introduction

It has been appreciated for a long time that the construction of consistent superstring theories requires appropriate projections on the underlying conformal field theory, most prominently the GSO projection. A careful implementation of these projections is e.g. required when one computes the massless spectrum of such theories. A clear conceptual understanding becomes even more mandatory when it comes to specifying boundary conditions for open strings. The analysis of the interplay between projections in superstring theories and boundary conditions for open strings is our main concern in this paper. We will concentrate on compactifications of type II superstring theories in which the internal part is an $N=2$ rational conformal field theory, among them in particular the Gepner models. Our approach enables us e.g. to derive formulas for Gepner model boundary states entirely from well-established principles. Where comparable with the literature, these results differ from the formulas obtained elsewhere except for some particularly simple models.

Let us be more explicit. In the construction of a superstring compactification one starts with specifying the vacuum configuration. This amounts to choosing a conformal field theory $\mathcal{C}_{\rm int}$ for the 'inner' or 'internal' sector. $\mathcal{C}_{\rm int}$ must satisfy a number of consistency constraints, such as possessing the correct Virasoro central charge, enough supersymmetry on the world sheet, and modular invariance. Afterwards, additional projections need to be imposed on $\mathcal{C}_{\rm int}$. This includes in particular the GSO projection, which ensures space-time supersymmetry. But there is another generic projection, too, which in the case of flat backgrounds looks quite innocent and which nevertheless will play an important role below. Namely, the total conformal field theory in question is a tensor product of the inner sector with the flat space-time part, and the constraint that necessitates a projection is that the spin structures for all fermionic fields on the world sheet must be aligned. In other words, the fermionic fields have to be either in the Ramond or in the Neveu–Schwarz sector simultaneously in each factor of the tensor product. As we will see, the interplay between these two projections has quite non-trivial consequences in non-flat backgrounds, in particular for the description of boundary states.

Now it is well-known that just projecting out states from a conformal field theory typically destroys its consistency, like e.g. modular invariance of the torus partition function. The projection therefore must be compensated by some additional manipulations, such as including new, twisted, sectors. For instance, in the case of the Gepner [1] construction, the inner sector conformal field theory $\mathcal{C}_{\rm int}$ one starts with can be written as a tensor product $\mathcal{C}_{k_1k_2\dots k_r} = \mathcal{C}_{k_1} \otimes \cdots \otimes \mathcal{C}_{k_r}$ of $N=2$ minimal models. On this theory $\mathcal{C}_{k_1k_2\dots k_r}$ one imposes fermion alignment and the GSO projection, but at the same time includes additional states that do not appear in the spectrum of the original tensor product theory [2]. Put differently, the torus partition function of the full Gepner conformal field theory $\mathcal{C}^{(\rm Gep)}$ – i.e. the theory that is obtained by these manipulations of projecting out old and of adding new states – contains non-diagonal ('twisted') contributions when viewed in terms of primary fields of the $N=2$ tensor product. In particular the vacuum field of the original theory $\mathcal{C}_{k_1k_2\dots k_r}$ not only gets combined with itself, but also with other fields – to be called *simple currents* – of the original theory. (The corresponding states play in fact a crucial role in the space-time physics. The associated vertex operators provide e.g. the gravitini.) At the chiral level, this means that the chiral symmetry algebra of the Gepner model is *extended* beyond that of the $N=2$ tensor product, namely precisely by including the relevant simple currents into the algebra [3]. Note that from the point of view of the extended chiral algebra, the partition function is diagonal. [1]

In short, the Gepner construction amounts to extending the underlying tensor product of $\mathcal{C}_{\text{int}}$ with the theory $\mathcal{C}_{\text{s-t}}$ that describes the surviving D flat non-compact space-time dimensions by certain simple currents. For concreteness we refer to this new theory $\mathcal{C}^{(\text{Gep})}$ as the *Gepner extension*. To separate generic aspects which are related to the space-time part from aspects that depend on the chosen inner sector it turns out to be simpler, and conceptually clearer, to break up the extension into two separate steps. Thus we first perform a suitable extension on the inner sector $\mathcal{C}_{\text{int}}$ alone. This way we arrive at a theory to which we refer as the *Calabi–Yau extension* $\mathcal{C}^{(\text{CY})}$. In models that possess a geometric interpretation as a sigma model on a Calabi–Yau manifold, it is this theory $\mathcal{C}^{(\text{CY})}$, rather than the original theory $\mathcal{C}_{\text{int}}$ (e.g. the mere tensor product $\mathcal{C}_{k_1 k_2 \dots k_r}$ of minimal models) that should be compared with the geometrical data. The proper combination of $\mathcal{C}^{(\text{CY})}$ with the space-time theory $\mathcal{C}_{\text{s-t}}$ then still requires further projections. Thus in a second step, we tensor $\mathcal{C}^{(\text{CY})}$ with $\mathcal{C}_{\text{s-t}}$, and thereafter perform yet another extension that involves both $\mathcal{C}^{(\text{CY})}$ and $\mathcal{C}_{\text{s-t}}$. As we will see, this latter extension is completely straightforward. This allows us to concentrate our attention on $\mathcal{C}^{(\text{CY})}$.

Simple current fields possess a variety of nice properties which allow for a very general and powerful treatment of arbitrary projections in which the chiral algebra gets enlarged [8–10]. Such simple current extensions have often been compared to orbifold constructions. For our purposes it is, however, indispensable not to mix up the two operations of simple current extension and orbifolding. While the respective closed string partition functions indeed display a certain similarity – both correspond to projecting out some states and adding new 'twisted' states – there is a significant difference at the level of the chiral symmetry algebras and, as a consequence, at the level of chiral conformal field theory. Briefly, in a simple current extension the chiral algebra $\mathfrak{A}$ gets *larger* [10] – the new algebra $\mathfrak{A}_{\text{ext}}$ consists of the old one plus the simple current fields – while in the orbifold construction it gets *smaller* [11] – the new algebra $\mathfrak{A}^G$ is the fixed point *sub*algebra of the old one with respect to the orbifold group G. Accordingly, a simple current extension of a given theory has, generically, fewer primary fields (inequivalent representations of the chiral algebra) than the original theory; the 'twisted states' that appear in the partition function correspond to left-right asymmetric combinations of ordinary $\mathfrak{A}$-representations. On the other hand, an orbifold has in general more primary fields than its mother theory, and the additional states correspond to new fields which appear already at the chiral level and carry 'twisted representations' of the original chiral algebra $\mathfrak{A}$. The differences between the chiral aspects of the two constructions become particularly relevant when it comes to the study of boundary effects. Still, these two types of constructing a new conformal field theory from a given one are closely related – they are in fact each other's inverse. The simple currents form an abelian group $\mathcal{G}$ under the fusion product, and it can be shown [12] that the operations of extension by a group $\mathcal{G}$ of simple currents and of taking the abelian orbifold with respect to the character group $G = \mathcal{G}^*$ are precisely inverse to each other. [2]

The reason for emphasizing the differences between the various extended theories that arise

[1] For simplicity, here we restrict ourselves to modular invariants of A-type for the $N = 2$ minimal models. Bulk spectra for other modular invariants have been computed in [3–5]. For a recent discussion of boundary states in bulk theories with modular invariants of D-type or E-type (based on the results of [6]), see [7].

[2] It follows e.g. that the sizes of the stabilizer groups of the simple current and of the orbifold action are complementary, i.e. full simple current orbits correspond to orbifold fixed points and vice versa. For more quantitative statements, see section 7 of [12].

in a string compactification rests in the following observation. Once one works with the appropriate conformal field theory $\mathcal{C}^{(\mathrm{CY})}$, the standard results for boundary conditions in (unitary) conformal field theories can be employed. In particular, Cardy's [13] construction of boundary states for boundary conditions that preserve the full chiral algebra can be applied directly.

The main points of this paper are the following. After establishing the necessary information about the Gepner and Calabi–Yau extensions (section 2), in section 3 we analyze in detail which symmetries of a Gepner model must be preserved and which ones can be broken by a given boundary condition. We also recall the recent increase of understanding of symmetry breaking boundary conditions (see [14, 15], and also [16] for applications to WZW models) and apply those results to Gepner models. We thereby obtain all boundary conditions that preserve full $N=2$ world sheet and half of space-time supersymmetry, the so-called A-type boundary conditions. This includes in particular the boundary states recently obtained in [17]. Our analysis reveals that within the boundary conditions of A-type, different 'automorphism types' appear, so that the A-type conditions can be naturally partitioned into several subsets. Boundary operators that change the automorphism type of the boundary conditions do not respect the GSO projection and therefore, generically, describe unstable brane configurations. Explicit formulas for all boundary states of Gepner models which preserve the full extended algebra are given.

In section 4 we turn to boundary conditions for which the action of the chiral algebra of the inner sector $\mathcal{C}_{\mathrm{int}}$ is twisted by some automorphism, in particular the B^{C}-type conditions which are based on the mirror automorphism of the $N=2$ superconformal algebra. As the relevant chiral algebra of the Gepner model is larger than that of $\mathcal{C}_{\mathrm{int}}$, it is necessary to lift the automorphism to the simple current extension $\mathcal{C}^{(\mathrm{CY})}$. In the analysis of both A- and B-type conditions we encounter the problem that such a lift is typically not unique; we employ arguments from quantum Galois theory to describe this non-uniqueness (more details are provided in the appendix). Finally, in section 5 we comment on the relationship between various "singular" structures encountered in Gepner models and their geometric counterparts and mention some open problems.

2 The Gepner extension and the Calabi–Yau extension

2.1 The bosonic string map

The simple current machinery was mostly developed for unitary conformal field theories. But since for maintaining the world sheet supersymmetry we must align the superghosts as well, the conformal field theory of our interest is definitely not unitary. To deal with this problem, we make use of the *bosonic string map* [18, 19, 1, 20]. This stratagem allows us to map the non-unitary chiral conformal field theory of our primary interest to another chiral conformal field theory that *is* unitary and that possesses the same topological data, i.e. modular matrices, but also braiding and fusing matrices. In particular, both in the open and closed string sector we can then work with ordinary partition functions rather than with supersymmetric partition functions. It is worthwhile to point out that while the bosonic string map was originally designed to construct heterotic theories, we use it here to simplify the description of type II superstring theories which are supersymmetric both in the left and the right chiral part.

Concretely, the fermions of the flat D-dimensional space-time theory $\mathcal{C}_{\mathrm{s-t}}$ together with the superghosts can be described by the lorentzian lattice $\mathrm{D}_{D/2,1}$; the first $D/2$ components come from the bosonization of the space-time fermions, and the last one (with opposite sign in the kinetic energy) from the bosonization of the superghost system. The bosonic string map B then amounts to replacing the non-unitary conformal field theory $\mathrm{D}_{D/2,1}$ by the unitary conformal field theory $\mathrm{D}_{D/2+3}$. Both of these theories have four primary fields, corresponding to the four conjugacy classes of the D-type simple Lie algebras; the map exchanges the characters for the zero (o) and vector (v) conjugacy classes and multiplies the characters for the spinor (s) and conjugate spinor (c) conjugacy classes by -1. Thus B is encoded in the matrix

$$B = \begin{pmatrix} 0 & 1\!\!1 & 0 & 0 \\ 1\!\!1 & 0 & 0 & 0 \\ 0 & 0 & -1\!\!1 & 0 \\ 0 & 0 & 0 & -1\!\!1 \end{pmatrix} , \tag{2.1}$$

where $1\!\!1$ is a unit matrix in the state space of the additional conformal field theory with which the theory for fermions and superghosts gets tensored, i.e. of the inner sector theory $\mathcal{C}_{\mathrm{int}}$ and the bosonic part of $\mathcal{C}_{\mathrm{s-t}}$. Denoting by a tilde quantities before the string map ('supersymmetric quantities'), and without tilde the ones after the string map ('ordinary CFT quantities'), we thus have, schematically, $\tilde{\chi} = B\chi$. For the modular transformation matrices this amounts to

$$\tilde{S} = B\, S\, B^{-1} , \qquad \tilde{T} = B\, T\, B^{-1} . \tag{2.2}$$

Given the modular invariant torus partition function Z of the ordinary conformal field theory, which satisfies $[Z,S]=0=[Z,T]$, it follows immediately that

$$\tilde{Z} := B\, Z\, B^{-1} \tag{2.3}$$

is modular invariant on the supersymmetric side.

The $N=2$ superconformal algebra contains a $\mathfrak{u}(1)$ current subalgebra. Via spectral flow, space-time supersymmetry is achieved when all $\mathfrak{u}(1)$ charges with respect to this subalgebra are odd integers. This condition can be fulfilled by a suitable projection on the allowed representations; this is precisely what the GSO projection does. Because of the exchange between o and v and the r-dependence of the $\mathfrak{u}(1)$ charge of D_r-spinors, the bosonic string map (2.1) changes all charges with respect to the $\mathfrak{u}(1)$ subalgebra of the $N=2$ algebra by 1 mod 2 [2]. This means in particular that while in the supersymmetric theory the GSO projection is to *odd* integral $\mathfrak{u}(1)$ charges, in the bosonic theory it is to *even* integral $\mathfrak{u}(1)$ charges.

2.2 Simple current extensions

Starting from the tensor product theory $\mathcal{C}_{\mathrm{s-t}} \otimes \mathcal{C}_{\mathrm{int}}$, the Gepner construction proceeds by projecting out certain states and adding new ones [2]. As already mentioned, technically this can be realized by the procedure of *simple current extension*. Basically, the simple current extension of a conformal field theory with chiral algebra $\mathfrak{A}$ by some group $\mathcal{G}$ of simple currents of integral conformal weight has the following effects [8, 10, 21].

- When fused with any other primary field λ of the theory, a simple current J yields just a single field Jλ. Thus a simple current is invertible in the fusion ring. The group $\mathcal{G}$ then acts on the fusion ring of the $\mathfrak{A}$-theory, and the simple current extension amounts to dividing out this action of $\mathcal{G}$.

▪ The *projection* amounts to keep only those fields λ which obey $Q_{\rm J}(\lambda)=0$ for all ${\rm J}\in\mathcal{G}$, where

$$Q_{\rm J}(\lambda) := \Delta_\lambda + \Delta_{\rm J} - \Delta_{{\rm J}\lambda} \bmod \mathbb{Z} = \Delta_\lambda - \Delta_{{\rm J}\lambda} \bmod \mathbb{Z} \tag{2.4}$$

is the so-called monodromy charge of the field λ of the $\mathfrak{A}$-theory with respect to the simple current (with integral conformal weight) ${\rm J}\in\mathcal{G}$.

▪ To obtain the primary fields of the extended theory we must organize the $\mathfrak{A}$-fields that survive the projection into orbits $[\lambda]$ under the fusion product with the currents in $\mathcal{G}$.

▪ The diagonal modular invariant of the extended theory reads

$$Z_{\rm ext} = \sum_{\substack{[\lambda] \\ Q_{\rm J}(\lambda)=0\ \forall {\rm J}\in\mathcal{G}}} |\mathcal{S}_\lambda| \, \Big| \sum_{{\rm J}\in\mathcal{G}/\mathcal{S}_\lambda} \chi_{{\rm J}\lambda}(\tau) \Big|^2 , \tag{2.5}$$

where $\mathcal{S}_\lambda \subseteq \mathcal{G}$ is the so-called stabilizer of λ, i.e. the subgroup $\mathcal{S}_\lambda$ consisting of those elements of $\mathcal{G}$ which leave λ fixed under the fusion product of the $\mathfrak{A}$-theory. Note that (2.5) is *non-diagonal* when viewed in terms of the primaries of the original theory. The terms of the form $\chi_\lambda \chi_{{\rm J}\lambda}^*$ indicate the inclusion of *twisted* states which are needed to ensure modular invariance. (For an analysis in the WZW case, see [22].) Also, both the stabilizer subgroup $\mathcal{S}_\lambda$ and the monodromy charge are well defined for orbits $[\lambda]$, not only for individual fields λ.

▪ When an orbit $[\lambda]$ has a non-trivial stabilizer, the factor of $|\mathcal{S}_\lambda|$ in the partition function (2.5) seems to indicate that the corresponding states occur several times. An additional 'quantum number' distinguishing those states is provided by a character of $\mathcal{S}_\lambda$, i.e. by $\psi_\lambda \in \mathcal{S}_\lambda^*$. Accordingly, the primary fields of the extended theory are completely labeled as

$$\lambda_{\rm ext} = [\lambda, \psi_\lambda] \,. \tag{2.6}$$

It is worth stressing that, while the prescription for projecting out states and adding new ones is already in itself sufficient for obtaining the spectrum of the model, to determine the complete modular properties of the model (and a fortiori for obtaining boundary conditions) it is indispensable to take proper care of such additional quantum numbers. Naively, in the case of a non-trivial stabilizer the projection rules appear to require the inclusion of the same state several times into the partition function. This would spoil unitarity of the modular S-matrix of the theory. The puzzle is resolved by realizing that those seemingly identical states are indeed distinguished by a further quantum number. Simple currents constitute a convenient conceptual framework for summarizing the required additional information.

▪ The modular S-matrix $S_{\rm ext}$ of the extended theory can be expressed in terms of the modular S-matrix S of the $\mathfrak{A}$-theory and of similar matrices $S^{\rm J}$ with ${\rm J}\in\mathcal{G}$. The latter describe the modular S-transformation of one-point chiral blocks (of the $\mathfrak{A}$-theory) on the torus with insertion J [21, 23]. Explicitly [21, 14], the matrix elements of $S_{\rm ext}$ (labeled, according to the above, by $\mathcal{G}$-orbits of monodromy charge zero $\mathfrak{A}$-primaries μ, supplemented by a character ψ_μ of the stabilizer $\mathcal{S}_\mu$) read

$$(S_{\rm ext})_{[\lambda,\psi_\lambda],[\mu,\psi_\mu]} = \frac{|\mathcal{G}|}{|\mathcal{S}_\lambda|\,|\mathcal{S}_\mu|} \sum_{{\rm J}\in\mathcal{S}_\lambda\cap\mathcal{S}_\mu} \psi_\lambda({\rm J})\, \psi_\mu({\rm J})^*\, S^{\rm J}_{\lambda,\mu} \,, \tag{2.7}$$

where $S^\Omega \equiv S$ is the ordinary S-matrix. When there are no fixed points (i.e., orbits with non-trivial stabilizer), then this expression collapses to

$$(S_{\rm ext})_{[\lambda],[\mu]} = |\mathcal{G}|\, S_{\lambda,\mu}\,, \tag{2.8}$$

so in this particular case the original S-matrix already contains all information about $S_{\rm ext}$.

Actually the formula (2.7) does not cover the most general situation. In full generality we rather have to account for the fact that the implementation of symmetries in quantum systems is typically only projective. This can also happen for the symmetries studied here. Quantitatively, the effect is described by a two-cocycle on the stabilizer group $\mathcal{S}_\mu$. What is remarkable is that this two-cocycle can be computed entirely in terms of the matrices $S^{\rm J}$. One can show [21] that the projectivity is properly taken into account by replacing $\mathcal{S}_\mu$ by the subgroup

$$\mathcal{U}_\mu \,\subseteq\, \mathcal{S}_\mu \tag{2.9}$$

of $\mathcal{S}_\mu$ on which the two-cocycle vanishes; $\mathcal{U}_\mu$ is called the untwisted stabilizer of μ.[3] In Gepner models with diagonal (or charge conjugation) torus partition function – the situation of our main interest below – one always has $\mathcal{U}_\mu = \mathcal{S}_\mu$, so that henceforth we will ignore this modification.

Via the Verlinde formula, the fusion rules of the extended theory can then be expressed through the fusion rules of the $\mathfrak{A}$-theory and the fixed point quantities $\mathcal{S}_\lambda$ and $S^{\rm J}$. For instance, for $\mathcal{G} = \mathbb{Z}_2 = \{\Omega, {\rm J}\}$ one finds

$$\begin{aligned}({\rm N}_{\rm ext})_{[\lambda,\psi],[\lambda',\psi']}^{\ \ [\lambda'',\psi'']} \;=\; \frac{1}{|\mathcal{S}_\lambda|\,|\mathcal{S}_{\lambda'}|\,|\mathcal{S}_{\lambda''}|}\Big[\, &{\rm N}_{\lambda,\lambda'}^{\ \ \lambda''} + {\rm N}_{\lambda,{\rm J}\lambda'}^{\ \ \lambda''} + 2 \sum_{\substack{\mu\\ {\rm J}\mu=\mu}} \Big(\psi'\psi''\, \frac{S_{\lambda,\mu} S^{\rm J}_{\lambda',\mu} S^{{\rm J}*}_{\lambda'',\mu}}{S_{\Omega,\mu}} \\ &+ \psi\psi''\, \frac{S^{\rm J}_{\lambda,\mu} S_{\lambda',\mu} S^{{\rm J}*}_{\lambda'',\mu}}{S_{\Omega,\mu}} + \psi\psi'\, \frac{S^{\rm J}_{\lambda,\mu} S^{\rm J}_{\lambda',\mu} S^{*}_{\lambda'',\mu}}{S_{\Omega,\mu}} \Big)\Big]\,. \end{aligned} \tag{2.10}$$

---

[3] The formula for the extended S-matrix then reads

$$(S_{\rm ext})_{[\lambda,\hat\psi_\lambda],[\mu,\hat\psi_\mu]} = |\mathcal{G}|\, [\, |\mathcal{S}_\lambda|\, |\mathcal{U}_\lambda|\, |\mathcal{S}_\mu|\, |\mathcal{U}_\mu|\,]^{-1/2} \sum_{{\rm J}\in\mathcal{U}_\lambda\cap\mathcal{U}_\mu} \hat\psi_\lambda({\rm J})\, \hat\psi_\mu({\rm J})^*\, S^{\rm J}_{\lambda,\mu}\,,$$

where $\hat\psi_\mu$ is a character of $\mathcal{U}_\mu \subseteq \mathcal{S}_\mu \subseteq \mathcal{G}$. Note that in this general situation even the labeling of primary fields is different from the case where $\mathcal{U}_\mu$ coincides with $\mathcal{S}_\mu$ for all μ; in place of the label (2.6) we now have $\lambda_{\rm ext} = [\lambda, \hat\psi_\lambda]$. Our notation for the simple current orbits is actually adapted to the general situation, as $\mathcal{G}_\mu/\mathcal{U}_\mu$ acts non-trivially on the characters $\hat\psi_\mu$ when $\mathcal{U}_\mu$ is a proper subgroup of $\mathcal{S}_\mu$.
Also, while the arguments in [21] were not sufficient to prove the formula rigorously, a proof is possible by combining them with the results of [24] on the uniqueness of the modularisation of a premodular category. Independently, various aspects of the formula can be tested directly [21, 23]. For instance, with the help of the computer program `kac` (see `http://norma.nikhef.nl/~t58/kac.html`) it was checked in a huge number of cases that it produces non-negative integers when inserted into the Verlinde formula. Moreover, manifestly the formula requires only information about the chiral conformal field theory, and even only information about topological aspects of the chiral theory. In particular it does not involve any knowledge about boundary conditions.
While in Gepner models with diagonal or charge conjugation invariant one always has $\mathcal{U}_\mu = \mathcal{S}_\mu$, for models where the extension of the $N=2$ tensor product is by a larger group – e.g. corresponding to taking non-diagonal modular invariants of the $N=2$ minimal models – cases where $\mathcal{U}_\mu$ is a proper subgroup of $\mathcal{S}_\mu$ can and do arise. This must e.g. be taken into account when analyzing the boundary states introduced in [7].

As is clear from (2.7), the rows and columns of the S^{J}-matrices are labeled by only those primaries of the $\mathfrak{A}$-theory that are fixed under J. Thus unless at least two of the fields λ, λ' and λ'' are fixed under J, the corresponding terms in (2.10) vanish.

It is worth emphasizing that a simple current extension amounts to nothing else than to a change of the underlying *chiral* conformal field theory. This fact, which is somewhat hidden in other treatments of projections (compare e.g. [25]), is of central importance for gaining a better understanding of boundary states in Gepner models. Namely, it implies in particular that all the features that are revealed in the analysis of these projections can be understood in a manner that is completely independent from our (yet uncomplete) understanding of world sheet boundary effects. Since the relevant results have undergone extensive physical and mathematical consistency checks which do not use any information about boundary conditions, we can safely exploit these structures as an input in the construction of boundary states for Gepner models.

A crucial property of superconformal field theories is that the supersymmetry automatically leads to the existence of certain simple currents. First of all, independently on the number of world sheet supersymmetries, every superconformal field theory has a distinguished simple current v: the generator of world sheet supersymmetry, which has order two and conformal dimension $\Delta = 3/2$. The monodromy charge with respect to v is 0 for primary fields in the Neveu–Schwarz sector and 1/2 for primaries in the Ramond sector. The 'superpartner' of a primary field λ is given by fusion product of λ with this simple current v.

In case the superconformal field theory has extended ($N = 2$) supersymmetry, there is yet another simple current s_{int}: the Ramond ground state R_0 with highest $\mathfrak{u}(1)$ charge. This can be seen after expressing the $\mathfrak{u}(1)$ current J of the $N = 2$ algebra in terms of a canonical free boson, $J(z) = \mathrm{i}\sqrt{c/3}\,\partial\phi(z)$. Then the Ramond ground state is given by $\exp(\mathrm{i}\sqrt{c/12}\,\phi)$. Its conformal dimension is $\Delta = c/24$, as befits a Ramond ground state, and it has the correct $\mathfrak{u}(1)$ charge $c/6$. In this formulation, it is easy to see that the monodromy charge with respect to this simple current equals half of the superconformal $\mathfrak{u}(1)$ charge of a field.

2.3 The Gepner extension

In this subsection we display the simple current extension that leads from an internal $N = 2$ superconformal theory $\mathcal{C}_{\mathrm{int}}$ to a consistent string background $\mathcal{C}^{(\mathrm{Gep})}$. Let us point out that this extension is not only applicable for the original Gepner models, but likewise for any other $N = 2$ compactification in which the internal theory is a rational conformal field theory, for instance for Kazama–Suzuki models.

The (flat) space-time bosons and Virasoro ghosts will play no role in what follows, and accordingly we suppress their contribution to $\mathcal{C}_{\mathrm{s-t}}$. What then remains of the space-time theory are the fermions and superghosts. After the bosonic string map, these are described by the level one WZW theory based on the Lie algebra $\mathrm{D}_{D/2+3}$; this (unitary) conformal field theory $\mathrm{D}_{D/2+3}$ has four primary fields, which we label as

$$\varpi \in \{o, s, v, c\}\,. \tag{2.11}$$

The Gepner extension is the extension of the tensor product theory $\mathrm{D}_{D/2+3} \otimes \mathcal{C}_{\mathrm{int}}$ by a certain simple current group $\mathcal{K}^{(\mathrm{Gep})}$, which accounts for fermion alignment and GSO projection. The group $\mathcal{K}^{(\mathrm{Gep})}$ is generated by some number r of order-two currents

$$\mathrm{v}_i := (v; v_{i,\mathrm{int}}) \tag{2.12}$$

together with

$$\mathrm{s}_{\mathrm{tot}} := (s; s_{\mathrm{int}})\,; \tag{2.13}$$

the fields v_i will be referred to as *alignment currents* (rather than as vector currents, as is done e.g. in [26]), and $\mathrm{s}_{\mathrm{tot}}$ as the *total spinor current*. When the inner sector $\mathcal{C}_{\mathrm{int}}$ can itself be written as a tensor product (e.g. of $N=2$ minimal models as in the original Gepner construction), then each tensor factor provides us with one of the currents $v_{i,\mathrm{int}}$, which then is a non-trivial field in the ith factor, tensored with the identity field of all other factors of $\mathcal{C}_{\mathrm{int}}$. The subspace with lowest conformal weight of the representation space of the bosonic part of the $N=2$ algebra corresponding to the field $v_{i,\mathrm{int}}$ has a dimension which is a multiple of two; it contains the two supercurrents $G_i^{\pm}$ of the ith tensor factor.

The order N_s of $\mathrm{s}_{\mathrm{tot}}$ is model dependent. Also, depending on the model the resulting group $\mathcal{K}^{(\mathrm{Gep})}$ is either the direct product of the $\mathbb{Z}_{N_s}$ generated by the total spinor current $\mathrm{s}_{\mathrm{tot}}$ and of the $\mathbb{Z}_2$ groups generated by the alignment currents v_i, or else some quotient of that direct product group. The latter happens when the field $(\mathrm{s}_{\mathrm{tot}})^{N_s/2}$ is contained in $(\mathbb{Z}_2)^r$, [4] and in that case the corresponding quotient is, as an abstract abelian group, the direct product of $\mathbb{Z}_{N_s}$ with $r{-}1$ copies of $\mathbb{Z}_2$. Thus the simple current group of the Gepner extension has the structure

$$\mathcal{K}^{(\mathrm{Gep})} = \mathbb{Z}_{N_s} \times (\mathbb{Z}_2)^{r-\eta} \quad \text{with} \quad \eta \in \{0,1\}\,. \tag{2.14}$$

The extension by the group generated by the currents v_i guarantees world sheet supersymmetry. Indeed, the total $N=2$ superconformal algebra must split into two modules of its bosonic subalgebra, the vacuum module and a module containing the supercurrents $G^{\pm} = \sum_{i=1}^{r} G_i^{\pm}$; this is so only *after* extension by the v_i. (Concretely, e.g. the two terms in $G_1^{\pm}{+}G_2^{\pm} \equiv G_1^{\pm}\otimes\mathbf{1} + \mathbf{1}\otimes G_2^{\pm}$ lie in two distinct irreducible modules of $\mathfrak{A}_{\mathrm{int}}$, and these modules get combined into an irreducible module of the extended algebra precisely due to the extension by $\mathrm{v}_1\mathrm{v}_2 = (o; v_{1,\mathrm{int}}v_{2,\mathrm{int}})$.) The extension by the total spinor current $\mathrm{s}_{\mathrm{tot}}$ implements the GSO projection and hence ensures space-time supersymmetry. The monodromy charge (2.4) with respect to the current $\mathrm{s}_{\mathrm{tot}}$ can be shown to coincide (modulo $\mathbb{Z}$) with half of the superconformal $\mathfrak{u}(1)$ charge of a state. Also, a change from the o to the v conjugacy class results in a change of this monodromy charge by $1/2 \bmod \mathbb{Z}$, and in the Ramond sector the same effect results from the r-dependence of the $\mathfrak{u}(1)$ charge of the spinors of D_r. Recalling the form (2.1) of the bosonic string map, we thus see again that it changes the effect of the GSO projection from a projection to odd integral $\mathfrak{u}(1)$ charges to a projection to even integral $\mathfrak{u}(1)$ charges.

Let us remark that the abelian orbifold construction that brings us back from the Gepner model $\mathcal{C}^{(\mathrm{Gep})}$ to the tensor product $\mathrm{D}_{D/2+3}\otimes\mathcal{C}_{\mathrm{int}}$ consists of orbifolding by the group $K^{(\mathrm{Gep})}$ that is generated by the automorphisms

$$\left.\begin{array}{l} \mathrm{v}_i \mapsto -\mathrm{v}_i\,, \quad \mathrm{v}_j \mapsto \mathrm{v}_j \ \ \text{for } j\neq i \\ \mathrm{s}_{\mathrm{tot}} \mapsto \mathrm{s}_{\mathrm{tot}}\,, \end{array}\right\} \text{ for } i=1,2,\dots,r\,, \qquad \text{and} \\ \mathrm{v}_j \mapsto \mathrm{v}_j \ \ \text{for all } j\,, \quad \mathrm{s}_{\mathrm{tot}} \mapsto \exp(2\pi\mathrm{i}/N_s)\,\mathrm{s}_{\mathrm{tot}}\,. \tag{2.15}$$

[4] For instance, in $N=2$ minimal models with odd level k, the spinor current s (see formula (3.25) below) satisfies $s^{2k+4}=v$. This results in the equality $\mathrm{s}_{\mathrm{tot}}^{N_s/2} = \prod_{i=1}^{r} \mathrm{v}_i$ when $\mathcal{C}_{\mathrm{int}}$ is the tensor product of r minimal models with only odd levels. For more details, see subsection 3.5.

Also note that the latter map corresponds to a shift $\phi \mapsto \phi + 4\pi\sqrt{3/c}/N_s$ of the free boson ϕ in terms of which the spinor current can be written as $\mathrm{s}_{\mathrm{tot}} = \exp(\mathrm{i}\sqrt{c/12}\,\phi)$.

2.4 The Calabi–Yau extension

The original Gepner construction – reformulated in the previous subsection in terms of simple current extensions – involves both the flat space-time part $\mathrm{D}_{D/2+3}$ and the inner sector $\mathcal{C}_{\mathrm{int}}$. This tends to obscure the connection with the geometric formulation in terms of sigma models on Calabi–Yau manifolds. The object in the Gepner model that corresponds to the compactification manifold in the geometric setting is not simply the conformal field theory $\mathcal{C}_{\mathrm{int}}$ – e.g. the tensor product $\mathcal{C}_{k_1 k_2 \dots k_r}$ of minimal models – but rather an extension of this tensor product, which will be specified shortly; we shall denote it by $\mathcal{C}^{(\mathrm{CY})}$ and call it the *Calabi–Yau extension*. The connection to geometric compactifications is usually derived using a Landau–Ginzburg description of the minimal models (for a different line of arguments, see [27]), and indeed this construction involves a non-trivial projection on the tensor product, commonly referred to as forming a Landau–Ginzburg orbifold [25]. In a second step, one combines this extended inner sector $\mathcal{C}^{(\mathrm{CY})}$ with flat space-time (i.e. tensors with the theory $\mathrm{D}_{D/2+3}$) and then performs an additional extension, which has more similarities to the GSO projection in ten flat dimensions. Unlike the step from $\mathcal{C}_{\mathrm{int}}$ to $\mathcal{C}^{(\mathrm{CY})}$, this further extension is completely straightforward. Let us stress that the procedure that we call the Calabi–Yau extension can be performed independently of any connection of the internal sector to a classical geometrical compactification.

Inspecting the Gepner extension, one observes that the group $\mathcal{K}^{(\mathrm{Gep})}$ contains many currents that have a trivial space-time part and therefore effectively define an extension of the internal theory $\mathcal{C}_{\mathrm{int}}$ alone. We denote the group of these currents by $\mathcal{K}^{(\mathrm{CY})}$. Using the fusion rules of the $\mathrm{D}_{D/2+3}$ theory (which are of the form $\mathbb{Z}_2 \times \mathbb{Z}_2$ when we compactify to $D = d+2 = 6$ dimensions, and $\mathbb{Z}_4$ for compactifications to $D = 4$ or 8), we find that $\mathcal{K}^{(\mathrm{CY})}$ is generated by all products of any two of the currents $v_{i,\mathrm{int}}$, and hence by

$$\mathrm{w}_i := v_{1,\mathrm{int}}\, v_{i,\mathrm{int}}\,, \qquad i \in \{2, 3, \dots, r\} \tag{2.16}$$

which will again be called alignment currents, together with the current [5]

$$\mathrm{u} := s_{\mathrm{int}}^2 (v_{1,\mathrm{int}})^{d/2}\,. \tag{2.17}$$

This group has the structure

$$\mathcal{K}^{(\mathrm{CY})} = \mathbb{Z}_{N_s/2} \times (\mathbb{Z}_2)^{r-1-\eta}\,, \tag{2.18}$$

where the contribution $\eta \in \{0, 1\}$ in the exponent accounts for the possibility that $\mathrm{u}^{N_s/4}$ is contained in the product of the $r-1$ $\mathbb{Z}_2$ groups that are generated by the currents w_i (compare the remarks before formula (2.14)).

The theory $\mathcal{C}^{(\mathrm{CY})}$ that is obtained upon extension of $\mathcal{C}_{\mathrm{int}}$ by $\mathcal{K}^{(\mathrm{CY})}$ inherits a simple current (sub)group $\{o^{(\mathrm{CY})}, s^{(\mathrm{CY})}, v^{(\mathrm{CY})}, c^{(\mathrm{CY})}\}$ that has the same fusion rules (i.e., $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$,

[5] The presence of the power of $v_{1,\mathrm{int}}$ accounts for the fact that $s^2 = v^{d/2}$ in the $\mathrm{D}_{D/2+3}$ theory. More explicitly, for compactifications to $D = 6$, the group $\mathcal{K}^{(\mathrm{CY})}$ contains all products of an even number of currents $v_{i,\mathrm{int}}$ and all products of $(s_{\mathrm{int}})^2$ with the former, while for compactifications to $D = 4$ or 8 in addition to all products of an even number of currents $v_{i,\mathrm{int}}$ one has all products of $(s_{\mathrm{int}})^2$ with an odd number of $v_{i,\mathrm{int}}$.

respectively) as the $\mathrm{D}_{D/2+3}$ theory. The final projections that bring us from $\mathrm{D}_{D/2+3} \otimes \mathcal{C}^{(\mathrm{CY})}$ to the Gepner extension $\mathcal{C}^{(\mathrm{Gep})}$ amount to extending by the simple current $(v; v^{(\mathrm{CY})})$, which aligns fermions in $\mathrm{D}_{D/2+3}$ and $\mathcal{C}^{(\mathrm{CY})}$, and in addition by either $(s; s^{(\mathrm{CY})})$ or $(c; s^{(\mathrm{CY})})$. This very last extension is the true analog of the GSO projection in flat space-time. In particular, when one combines the left and right halves of the theory, the choice between type IIA and IIB theories is equivalent to a choice between the extension by $(s; s^{(\mathrm{CY})})$ both on the left and the right (or equivalently, by $(c; s^{(\mathrm{CY})})$ on both sides), or else by $(s; s^{(\mathrm{CY})})$ on one side and by $(c; s^{(\mathrm{CY})})$ on the other. We remark that all currents in the extension of $\mathrm{D}_{D/2+3} \otimes \mathcal{C}^{(\mathrm{CY})}$ to $\mathcal{C}^{(\mathrm{Gep})}$ act freely. Thus the simple formula (2.8) for the modular S-matrix of the extension applies, and hence as announced this extension is straightforward.

The dependence of the precise structure of the $\mathcal{K}^{(\mathrm{CY})}$ extension on the number of compactified dimensions can be traced back to the fact that the internal spectral flow operator, mapping the R to the NS sector, changes the $\mathfrak{u}(1)$ charge by $c_{\mathrm{int}}/6$, where $c_{\mathrm{int}} = 12 - 3d/2$ is the central charge of $\mathcal{C}_{\mathrm{int}}$. Thus, while in the NS sector we always project onto integral internal $\mathfrak{u}(1)$ charge, the internal $\mathfrak{u}(1)$ charges in the R sector are either integers or in $\mathbb{Z}+1/2$, depending on the external dimension being 6 or $4, 8$, respectively. The integrality of the $\mathfrak{u}(1)$ charges in the NS sector is necessary to make a correspondence between chiral primary fields and differential forms in the geometric compactification possible, and is therefore highly welcome. In fact, s_{int}^2 can be identified model independently as the square of the Ramond ground state with maximal $\mathfrak{u}(1)$ charge.

As we shall see below, there is yet another intermediate theory, to be denoted by $\mathcal{C}_{\mathrm{wsusy}}$, that is of interest. This is the theory that one obtains from the inner sector $\mathcal{C}_{\mathrm{int}}$ by extending with the subgroup $(\mathbb{Z}_2)^{r-1} \subset \mathcal{K}^{(\mathrm{CY})}$ generated by the alignment currents w_i only, i.e. by enforcing only fermion alignment, and hence world sheet supersymmetry, on the internal theory. $\mathcal{C}_{\mathrm{wsusy}}$ will play an important role in the analysis of boundary conditions. We summarize the relation between the various extensions schematically as

$$\mathrm{D}_{D/2+3} \otimes \mathcal{C}_{\mathrm{int}} \prec \mathrm{D}_{D/2+3} \otimes \mathcal{C}_{\mathrm{wsusy}} \prec \mathrm{D}_{D/2+3} \otimes \mathcal{C}^{(\mathrm{CY})} \prec \mathcal{C}^{(\mathrm{Gep})} . \tag{2.19}$$

Note that the group that furnishes the extension from $\mathcal{C}_{\mathrm{wsusy}}$ to $\mathcal{C}^{(\mathrm{CY})}$ is the cyclic group generated by the image U of the simple current u (3.24) in the $\mathcal{C}_{\mathrm{wsusy}}$-theory. The order of U can differ from the order $N_s/2$ of u by a factor of two; it is given by $N_s'/2$ with

$$N_s' = 2^{-\eta} N_s , \tag{2.20}$$

where η is the integer introduced in formula (2.14).

We can – and will – simplify the discussion and restrict our attention in the sequel to the intermediate theory $\mathcal{C}^{(\mathrm{CY})}$ (and later on also to $\mathcal{C}_{\mathrm{wsusy}}$). As the additional extension to $\mathcal{C}^{(\mathrm{Gep})}$ is so simple, we do not loose any essential features when doing so. In particular the issue of fixed points arises always only in the study of $\mathcal{C}^{(\mathrm{CY})}$. To conclude this section, let us emphasize that the construction described above is model independent and does not rely on specific aspects of the $N = 2$ superconformal field theory used in the inner sector.

3 A-type boundary conditions

3.1 Intermediate chiral algebras

Already in closed string theory the simple current extension leading to $\mathcal{C}^{(\mathrm{CY})}$ must be taken into account properly. In particular, a careful treatment of fixed points is compulsory to find the correct massless spectrum, compare e.g. [26, 28]. When open strings are present, for the following reason an even deeper understanding of fixed point resolution is required. In the computation of the massless spectrum one just counts states, thus a detailed understanding of the underlying partition functions suffices. In contrast, as was first realized by Cardy [13], in the construction of boundary states the modular S-matrix enters directly; therefore a complete knowledge of this matrix in simple current extensions is required as well. Moreover, open string partition functions (annulus amplitudes) also implicitly contain the modular matrices, so that not even the open string *spectrum* can be obtained correctly without proper resolution of the fixed points. In this context, it is an important observation that typically the Calabi–Yau extension, and hence also the Gepner extension, do possess fixed points. For instance, as will be discussed in subsection 3.5, in the case of a tensor product $\mathcal{C}_{\mathrm{int}}=\mathcal{C}_{k_1 k_2 \ldots k_r}$ of $N=2$ minimal models, fixed points arise precisely if at least one level k_i is even.

The group of automorphisms of the $N=2$ superconformal algebra is the Lie group O(2). This group has two connected components, and any element of the component not connected to the identity can be obtained by composing an element of the identity component with the mirror automorphism (see formula (4.2)). Those automorphisms of the chiral algebra of a tensor product of internal models that respect the $N=2$ structure are generically given by O(2) as well (but additional permutation symmetries are present when some of the factors of the tensor product are identical). Accordingly, in such string compactifications one conventionally distinguishes between two classes of boundary states: Those which correspond to an automorphism in the identity component of O(2), and those corresponding to an automorphism in the other component. In the literature, the former states are often collectively referred to as A-*type boundary states*, while latter are said to be of B-*type*. A-type boundary conditions leave the chiral algebra $\mathfrak{A}_{\mathrm{int}}$ of the inner sector $\mathcal{C}_{\mathrm{int}}$ invariant; insisting that also an $N=2$ subalgebra of the extension $\mathcal{C}^{(\mathrm{CY})}$ remains unbroken, the A-type automorphism must be the identity map. In this section we study in detail boundary conditions $|a\rangle_A^{(\mathrm{CY})}$ which do preserve $\mathfrak{A}_{\mathrm{int}}$, i.e. which satisfy

$$\big(Y_n \otimes \mathbf{1} + (-1)^{\Delta_Y-1}\, \mathbf{1} \otimes Y_{-n}\big)|a\rangle_A^{(\mathrm{CY})} = 0 \tag{3.1}$$

for every field $Y(z)=\sum_{n\in\mathbb{Z}} Y_n z^{-n-\Delta_Y}$ of conformal weight Δ_Y in $\mathfrak{A}_{\mathrm{int}}$.

We start our analysis by recalling that the total chiral algebra $\mathfrak{A}^{(\mathrm{CY})}$ of the Calabi–Yau extension $\mathcal{C}^{(\mathrm{CY})}$ is obtained from $\mathfrak{A}_{\mathrm{int}}$ by a simple current extension with the group $\mathcal{K}^{(\mathrm{CY})}$. A generic boundary state $|a\rangle_A^{(\mathrm{CY})}$ will not preserve all of $\mathfrak{A}^{(\mathrm{CY})}$, but only some subalgebra $\mathfrak{A}_a$ containing $\mathfrak{A}_{\mathrm{int}}$. This subalgebra cannot be arbitrary, though. First of all, we are interested in conformally invariant boundary conditions only, and hence the Virasoro subalgebra of $\mathfrak{A}^{(\mathrm{CY})}$ must be preserved. This is automatically satisfied by the boundary states $|a\rangle_A^{(\mathrm{CY})}$, since the Virasoro algebra is already contained in the inner sector algebra $\mathfrak{A}_{\mathrm{int}}$. But in addition the preserved subalgebra must have enough structure to allow for the construction of conformal blocks and, based on them, of correlation functions. One therefore has to require that $\mathfrak{A}_a$ must again be a vertex operator algebra. To get an overview over all boundary conditions of our present interest, we thus look for all vertex operator algebras $\mathfrak{A}$ that lie between $\mathfrak{A}_{\rm int}$ and $\mathfrak{A}^{\rm (CY)}$:

$$\mathfrak{A}_{\rm int} \subseteq \mathfrak{A} \subseteq \mathfrak{A}^{\rm (CY)} . \tag{3.2}$$

For general vertex operator algebras, this classification of subalgebras would be a hopeless problem. Here, however, we know that the inner sector chiral algebra $\mathfrak{A}_{\rm int}$ can be characterized as the subalgebra of $\mathfrak{A}^{\rm (CY)}$ that is left pointwise fixed by the group $K^{\rm (CY)}$ of automorphisms of $\mathfrak{A}^{\rm (CY)}$, described in formula (2.15), which is dual to $\mathcal{K}^{\rm (CY)}$. This observation allows us to employ a basic result from the Galois theory for vertex operator algebras [29–31], which tells us that the possible chiral algebras between $\mathfrak{A}_{\rm int}$ and $\mathfrak{A}^{\rm (CY)}$ are in one-to-one correspondence with subgroups of the group $K^{\rm (CY)}$. We conclude that to every boundary state $|a\rangle_A^{\rm (CY)}$ of $\mathcal{C}^{\rm (CY)}$ we can associate a subgroup $K_a^{\rm (CY)}$ of $K^{\rm (CY)}$, such that the subalgebra of $\mathfrak{A}^{\rm (CY)}$ that is preserved by the boundary condition is the fixed point algebra with respect to $K_a^{\rm (CY)}$.

3.2 Boundary states and automorphism types

Boundary conditions that preserve a fixed point algebra of the bulk symmetries with respect to some finite abelian group of automorphisms have been studied in [15, 12]. When applied to the present situation, the pertinent results of [15, 12] may be summarized as follows.

- Using notation from the unextended theory $\mathcal{C}_{\rm int}$, the boundary states can be labeled in a way much similar to the labeling (2.6) of primary fields of the extended theory $\mathcal{C}^{\rm (CY)}$, namely as

$$a = [\mu, \psi_\mu] . \tag{3.3}$$

The difference is that, unlike in (2.6), here μ can be any primary field label of $\mathcal{C}_{\rm int}$, i.e. now there is no restriction on the monodromy charge. [6]

- When $K_a^{\rm (CY)}$ is non-trivial, then $|a\rangle_A^{\rm (CY)}$ can no longer be written as a linear combination of Ishibashi states of $\mathcal{C}^{\rm (CY)}$. This is simply due to the fact that the preserved chiral symmetry is then not big enough to guarantee that all $\mathfrak{A}^{\rm (CY)}$-descendants are reflected at the boundary in the same way, so that different descendants must be treated differently. The boundary state can, however, still be expressed in terms of suitable generalizations $|\cdots\rangle\!\rangle$ of the Ishibashi states of the unextended theory $\mathcal{C}_{\rm int}$. These states are labeled by a pair (λ, ψ_λ), where now again the restriction of zero monodromy charge is to be imposed on the primary field λ (but no simple current orbit is taken any longer). [7]

Heuristically the situation can be understood as follows. On the side of Ishibashi states, only states with vanishing monodromy charge are present; in the orbifold language of [25], only states in the untwisted sector appear. This means that the orbifold element in 'space' direction on the torus is always trivial, whereas we still project, i.e. we still deal with non-trivial group elements in 'time' direction. According to Cardy's ideas, the boundary states are obtained by a modular S-transformation from the Ishibashi states. After that transformation we only have

[6] At this point, the possibility of having untwisted stabilizers $\mathcal{U}_\mu \subset \mathcal{S}_\mu$ (see formula (2.9)) must in general be taken into account. Then ψ_μ gets in fact replaced by a character $\hat\psi_\mu$ of the untwisted stabilizer $\mathcal{U}_\mu$, and the simple current orbit is obtained by an action that also changes $\hat\psi_\mu$ in a non-trivial way. Also, the prefactor of $\tilde S$ gets changed analogously as in the formula in footnote 3.

[7] Here ψ_λ is always a character of the *full* stabilizer, even when the untwisted stabilizer is strictly smaller than the stabilizer [15].

the trivial group element in 'time' direction, which implies that there is no projection, so that orbits appear. However, in 'space' direction we now have non-trivial elements, and therefore the twisted sectors show up in the description of the boundary states.

To make these heuristic ideas quantitative, we introduce a matrix $\tilde S$ that takes over the role that the usual S-matrix plays in the Cardy case. As shown in [15, 12], such a matrix indeed exists. The following structures were uncovered.

- The expansion of the boundary state $|a\rangle_A^{(\mathrm{CY})}$ with respect to the generalized Ishibashi states reads

$$|[\mu,\psi_\mu]\rangle_A^{(\mathrm{CY})} = \sum_{\lambda,\psi_\lambda} \frac{\tilde S_{(\lambda,\psi_\lambda),[\mu,\psi_\mu]}}{\sqrt{\tilde S_{(\Omega),[\mu,\psi_\mu]}}}\, |(\lambda,\psi_\lambda)\rangle\!\rangle\,. \tag{3.4}$$

The matrix $\tilde S$ appearing here can be expressed as

$$\tilde S_{(\lambda,\psi_\lambda),[\mu,\psi_\mu]} = \frac{|\mathcal G|}{|\mathcal S_\lambda|\,|\mathcal S_\mu|} \sum_{\mathrm J\in\mathcal S_\lambda\cap\mathcal S_\mu} \psi_\lambda(\mathrm J)\,\psi_\mu(\mathrm J)^*\, S^{\mathrm J}_{\lambda,\mu}\,, \tag{3.5}$$

which is similar to (2.7) (but remember that now also twisted sectors are allowed in the second label). Complete information on the boundary state, like brane tensions or RR charges, is encoded in this matrix $\tilde S$.

- Note the similarity between the result (3.4) and Cardy's formula [13] for symmetry preserving boundary conditions, in which the modular S-matrix appears in place of $\tilde S$. It turns out that the matrix $\tilde S$ has still more in common with the modular S-matrix. Indeed, as first realized in [32,33], a subset of the sewing constraints [34] for correlation functions of a rational conformal field theory can be isolated which leads to a simple non-linear equation for the bulk-boundary coefficients for excitation of the vacuum field on the boundary. As pointed out in [35], this equation means that the reflection coefficients constitute one-dimensional representations of a certain finite-dimensional associative algebra, which generalizes the fusion rule algebra and is called the *classifying algebra.* [8]

These results allow us to introduce the notion of an *elementary boundary condition*; this furnishes by definition an *irreducible* representation of the classifying algebra. Thus the elementary boundary conditions are in one-to-one correspondence with the one-dimensional irreducible representations of the classifying algebra. The matrix $\tilde S$ as given by (3.5) diagonalizes the structure constants of the classifying algebra, analogously as the modular S-matrix diagonalizes the fusion rules. In string theory, on the other hand, one must in addition introduce (Chan–Paton) multiplicities for boundary conditions. Thus the space of all boundary conditions to be considered in string theory forms a cone over the elementary boundary conditions, and generically one is dealing with higher-dimensional, and hence necessarily reducible, representations of the classifying algebra. It is also quite common that some of the solutions that are present as elementary boundary conditions in the conformal field theory possess a zero Chan–Paton multiplicity, i.e. do not appear at the string theory level at all.

Based on the properties of the matrix $\tilde S$, the space of boundary conditions for $\mathfrak A^{(\mathrm{CY})}$ that preserve $\mathfrak A_{\mathrm{int}}$ can be analyzed in detail [15,12]. One important result is that each of the boundary states studied here possesses a definite *automorphism type*. This means that it can be written

[8] Extending Cardy's [13] work, it was shown in [12] that the structure constants of this algebra are traces on suitable spaces of conformal blocks.

as a linear combination of *twisted Ishibashi states*, where the twist is a fixed automorphism ω of $\mathfrak{A}^{(\mathrm{CY})}$. Such a twisted Ishibashi state $|[\lambda,\psi_\lambda]\rangle\!\rangle_\omega$ obeys the twisted Ward identity

$$\big(\mathcal{Y}_n \otimes \mathbf{1} + (-1)^{\Delta_{\mathcal{Y}}-1}\, \mathbf{1} \otimes \omega(\mathcal{Y}_{-n})\big)\, |[\lambda,\psi_\lambda]\rangle\!\rangle_\omega = 0 \tag{3.6}$$

for every field $\mathcal{Y}$ of conformal weight $\Delta_{\mathcal{Y}}$ in $\mathfrak{A}^{(\mathrm{CY})}$. (This generalizes the usual definition of Ishibashi states of Dirichlet-type for the free boson. In the terminology of [36, 37], the relation (3.6) says that ω provides the *glueing condition* of the boundary condition.)

The subset of A_ω-*type boundary states* – that is, of all boundary states which preserve $\mathfrak{A}_{\mathrm{int}}$ and which have some prescribed automorphism type ω – corresponds to a subalgebra of the classifying algebra. The structure constants of this subalgebra can be expressed in terms of traces of the action of ω on the space of the relevant three-point conformal blocks. This again generalizes the Cardy case, in which the structure constants are just the fusion rules, which in turn are nothing else than traces of the identity automorphism on spaces of three-point blocks.

The fact that the boundary states that preserve $\mathfrak{A}_{\mathrm{int}}$ come in several different automorphism types can also be understood as follows. The identity map on $\mathfrak{A}_{\mathrm{int}}$ can be lifted to an automorphism of $\mathfrak{A}^{(\mathrm{CY})}$ in several distinct ways. The group of these lifts of automorphisms is just the quantum Galois group for the extension. Each element of this group gives us an automorphism type. Let us stress that the fact that all the boundary conditions that are commonly referred to as A-type do possess an automorphism type is non-trivial indeed. It reflects the equally non-trivial statement of quantum Galois theory that all intermediate algebras are obtained as fixed algebras.

Once the boundary states are given explicitly, i.e. once the matrix $\tilde S$ is known, all annulus amplitudes can be computed by sandwiching a string propagator $q^{L_0+\tilde L_0-c/12}$ between the two appropriate boundary states. But already from the general expression (3.5) above (and from general properties of the matrices S^{J}), it can be established via representation theoretic arguments [15] that in full generality the annulus coefficients are non-negative integers, as befits the coefficients of the open string partition function. The completeness and associativity properties [33] of the annulus coefficients can be shown to be satisfied as well. Further, one can write the annulus amplitude for two boundary conditions of automorphism types ω_1 and ω_2 as a sum of characters $\chi'_{[\nu,\psi_\nu]'}$ of the extension of $\mathcal{C}_{\mathrm{int}}$ by the subgroup

$$\mathcal{K}_{\omega_1\omega_2} := \{\mathrm{J} \in \mathcal{K}^{(\mathrm{CY})} \,|\, Q_{\mathrm{J}}(\omega_1) = 0 = Q_{\mathrm{J}}(\omega_2)\} \tag{3.7}$$

of $\mathcal{K}^{(\mathrm{CY})}$. (Here the isomorphism between $K^{(\mathrm{CY})}$ and the dual group $(\mathcal{K}^{(\mathrm{CY})})^*$ is used, i.e. the automorphisms are regarded as characters of $\mathcal{K}^{(\mathrm{CY})}$.) The coefficients then read [15] [9]

$$\mathrm{A}_{[\mu_1,\psi_1],[\mu_2,\psi_2]}^{[\nu,\psi_\nu]'} = \frac{|\mathcal{K}_{\omega_1\omega_2}|}{|\mathcal{K}^{(\mathrm{CY})}|} \sum_{[\lambda,\psi'_\lambda]'} \frac{|\mathcal{S}_\lambda|}{|\mathcal{S}'_\lambda|} \sum_{\psi_\lambda \succ \psi'_\lambda} \tilde S^*_{(\lambda,\psi_\lambda),[\mu_1,\psi_1]} \tilde S_{(\lambda,\psi_\lambda),[\mu_2,\psi_2]} S'_{[\lambda,\psi'_\lambda]',[\nu,\psi_\nu]'} / S'_{[\lambda,\psi'_\lambda]',[\Omega]'}\,, \tag{3.8}$$

where S' is the modular S-matrix of the $\mathcal{K}_{\omega_1\omega_2}$-extension of $\mathcal{C}_{\mathrm{int}}$, and where the second summation is over all $\mathcal{S}_\lambda$-characters ψ_λ that restrict to the given character ψ'_λ of the subgroup $\mathcal{S}'_\lambda = \mathcal{S}_\lambda \cap \mathcal{K}_{\omega_1\omega_2}$.

[9] Here again we suppress the changes that arise when genuine untwisted stabilizer groups are present; see formula (6.23) of [15].

Another important conclusion to be drawn is that the monodromy charge constitutes a grading of the annulus coefficients, in the following sense. It follows from the result (3.8) that

$$\mathrm{A}^{[\nu,\psi_\nu]'}_{[\mu_1,\psi_1],[\mu_2,\psi_2]} = \mathrm{e}^{2\pi\mathrm{i}[Q_\mathrm{J}(\mu_2)-Q_\mathrm{J}(\mu_1)+Q_\mathrm{J}(\nu)]} \cdot \mathrm{A}^{[\nu,\psi_\nu]'}_{[\mu_1,\psi_1],[\mu_2,\psi_2]}\,, \tag{3.9}$$

so that the annulus coefficient $\mathrm{A}^{[\nu,\psi_\nu]'}_{[\mu_1,\psi_1],[\mu_2,\psi_2]}$ vanishes unless $Q_\mathrm{J}(\nu)=Q_\mathrm{J}(\mu_1)-Q_\mathrm{J}(\mu_2)$ for all $\mathrm{J}\in\mathcal{K}^{(\mathrm{CY})}$. Thus all open string states that appear in the annulus amplitude for two boundary conditions of automorphism types ω_1 and ω_2 have a common monodromy charge Q_J with respect to any current J in $\mathcal{K}^{(\mathrm{CY})}$, and this common value is given by

$$\exp(2\pi\mathrm{i}Q_\mathrm{J}) = \omega_1^{-1}\omega_2(\mathrm{J})\,. \tag{3.10}$$

3.3 World sheet supersymmetry and space-time supersymmetry

We now analyze what boundary conditions in string compactifications of Gepner type preserve the (super)symmetries that have to be imposed to obtain a consistent bulk theory. To this end we study in more detail the twisted Ward identities (3.6). Again we start our discussion with boundary states that preserve the chiral symmetry algebra $\mathfrak{A}_\mathrm{int}$ of the inner sector $\mathcal{C}_\mathrm{int}$. As already mentioned, such boundary states are often collectively said to be of A-type, see for instance [37–40]. However, as we have detailed above, the inner sector $\mathcal{C}_\mathrm{int}$ of the string compactification needs to be extended to $\mathcal{C}^{(\mathrm{CY})}$, which does not have $\mathfrak{A}_\mathrm{int}$ as chiral symmetry, but rather its simple current extension $\mathfrak{A}^{(\mathrm{CY})}$. To identify boundary states that are relevant for string theory, in particular those that can be given a geometric interpretation as D-branes, it is necessary to understand what part of the extended algebra $\mathfrak{A}^{(\mathrm{CY})}$ is preserved or broken by a given boundary state.

Recall that the simple current extension from $\mathcal{C}_\mathrm{int}$ to $\mathcal{C}^{(\mathrm{CY})}$ consists of a part that ensures world sheet supersymmetry and another part necessary for space-time supersymmetry. Accordingly, the automorphism type ω_a of a boundary condition that preserves $\mathfrak{A}_\mathrm{int}$ carries information concerning both the extension by the alignment currents $\mathrm{w}_i=v_{1,\mathrm{int}}v_{i,\mathrm{int}}$ $(i=2,3,\dots,r)$ and the extension by $\mathrm{u}=s_\mathrm{int}^2 v_{1,\mathrm{int}}^{d/2}$. These individual pieces of information can be thought of as measuring which supersymmetries are broken or conserved by the boundary state. More concretely, we can attribute to every boundary state $|a\rangle_A^{(\mathrm{CY})}$ an element ω_a of the orbifold group $K^{(\mathrm{CY})}$ that is dual to $\mathcal{K}^{(\mathrm{CY})}$. As an abstract abelian group, this is again $\mathbb{Z}_{N_s/2}\times\mathbb{Z}_2^{r-1-\eta}$. The automorphism ω_a occurs in the twisted Ward identities (3.6) satisfied by $|a\rangle_A^{(\mathrm{CY})}$; explicitly, we have

$$\begin{aligned} \omega_a(\mathrm{w}_i) &= \zeta_{a,i}\,\mathrm{w}_i \quad \text{with} \quad \zeta_{a,i}\in\{\pm1\}\,, \\ \omega_a(\mathrm{u}) &= \mathrm{e}^{2\mathrm{i}\vartheta_a}\,\mathrm{u} \quad \text{with} \quad 2\vartheta_a\in 2\pi\mathbb{Z}/(N_s/2)\,, \end{aligned} \tag{3.11}$$

and hence

$$\begin{aligned} &\big((\mathrm{w}_i)_n\otimes\mathbf{1} + \zeta_{a,i}\,(-1)^{\Delta_{\mathrm{w}_i}-1}\,\mathbf{1}\otimes(\mathrm{w}_i)_{-n}\big)\,|a\rangle_A^{(\mathrm{CY})} = 0\,, \\ &\big(\mathrm{u}_n\otimes\mathbf{1} + \mathrm{e}^{2\mathrm{i}\vartheta_a}\,(-1)^{\Delta_\mathrm{u}-1}\,\mathbf{1}\otimes\mathrm{u}_{-n}\big)\,|a\rangle_A^{(\mathrm{CY})} = 0\,. \end{aligned} \tag{3.12}$$

Also recall that for each tensor factor of the internal $N=2$ theory the field $v_{i,\mathrm{int}}$ contains the supersymmetry charges $G_i^\pm$. Now $N=1$ *world sheet supersymmetry* plays the role of a gauge symmetry of perturbative superstring theory. In order not to destroy this constitutive feature of superstrings, $N=1$ world sheet supersymmetry must not be broken by any boundary state that is present (i.e., has a non-zero Chan–Paton multiplicity) at the string theory level. Concretely, we have the relation $\{Q_{\rm BRST},\beta\}=G$ between the $N=1$ supercurrent G, the BRST charge $Q_{\rm BRST}$ and the superghost β. When combined with the identities

$$(Q_{\rm BRST}\otimes\mathbf{1}-\mathbf{1}\otimes Q_{\rm BRST})\,|a\rangle_A^{(\rm CY)}=0 \qquad\text{and}\qquad (\beta_r\otimes\mathbf{1}+\mathrm{i}\epsilon\,\mathbf{1}\otimes\beta_{-r})\,|a\rangle_A^{(\rm CY)}=0 \tag{3.13}$$

($\epsilon\in\{\pm1\}$) which encode BRST invariance of the boundary state and the boundary condition for the superghost (which is model independent and independent of the chosen boundary condition for the theory $\mathcal{C}^{(\rm CY)}$), this relation implies that we must also have

$$(G_r\otimes\mathbf{1}+\mathrm{i}\epsilon\,\mathbf{1}\otimes G_{-r})\,|a\rangle_A^{(\rm CY)}=0\,. \tag{3.14}$$

Comparison of this Ward identity with the result (3.12) then tells us that we must require the invariance property $\omega_a(G)=G$ of the $N=1$ supercurrent.

The extended theory $\mathcal{C}^{(\rm CY)}$ actually possesses $N=2$ world sheet supersymmetry, with supercurrents $G^\pm=\sum_i G_i^\pm$, into which the $N=1$ supersymmetry can be embedded in several different ways, namely as $G=(\mathrm{e}^{\mathrm{i}\gamma}G^+ + \mathrm{e}^{-\mathrm{i}\gamma}G^-)/\sqrt2$ for any $\gamma\in\mathbb{R}$. By inspecting the operator products of the supercurrents, it follows that there exists *some* $N=1$ subalgebra that is preserved if and only if in the relations (3.11) we have $\zeta_{a,i}=1$ for all i, i.e. if and only if the part of the automorphism type that concerns the alignment currents is trivial. (This appears to have been ignored in part of the literature.) As a matter of fact, in that case the boundary condition preserves *all* $N=1$ subalgebras, and hence even the whole $N=2$ algebra.

Thus from now on we only admit those automorphism types which obey

$$\omega_a(\mathrm{w}_i)=\mathrm{w}_i\,. \tag{3.15}$$

(Put differently, in the string theory any conformal field theory boundary condition which violates this relation is assigned Chan–Paton multiplicity zero.) According to the Ward identities (3.12), the automorphism type of any of the remaining boundary conditions is then completely characterized by a single number. This number is essentially given by $\vartheta\in2\pi\mathbb{Z}/N_s$; but we also have to take into account that the order $N_s'/2$ of the image U of u in the theory $\mathcal{C}_{\rm wsusy}$ can be different from the order $N_s/2$ of u itself. Thus the boundary conditions must rather be labeled by

$$\theta=2\pi n/N_s' \quad\text{with}\quad n\in\{0,1,\dots,N_s'{-}1\}\,. \tag{3.16}$$

We shall denote the corresponding automorphism type of branes by A_θ. Note that here we use θ instead of 2θ as label, even though from the point of view of the Calabi–Yau extension, the automorphism types θ and $\theta{+}\pi$ cannot be distinguished. We do so because in full string theory (i.e., for $\mathcal{C}^{(\rm Gep)}$ rather than $\mathcal{C}^{(\rm CY)}$), the two automorphism types differ on the final GSO projection.

This restriction to boundary conditions that preserve not only $\mathfrak{A}_{\rm int}$, but even its extension by the alignment currents i.e. the algebra $\mathfrak{A}_{\rm wsusy}$, also ensures that the annulus partition function between any two boundary conditions has an open string spectrum with world sheet supersymmetry.

In contrast to world sheet supersymmetry, *space-time supersymmetry* is not an indispensable ingredient of a superstring theory. Also, it is not determined by chiral considerations alone, in the sense that the space-time supersymmetry generators Q are given by closed string operators in which left- and right-movers are combined. Indeed, they arise as zero modes

$$Q = \int \mathrm{d}^2 z \; s(z, \bar{z}) \tag{3.17}$$

of fields in the full conformal field theory obtained by putting together both chiral halves. In an $N=2$ string compactification, the field $s(z,\bar{z})$ consists of a spectral flow operator on one chiral half combined with the vacuum field of the other chiral half (for more explicit expressions, see e.g. [2, 40]). Thus there are both left- and right-moving supersymmetry charges, Q_L and Q_R. In the context of open strings, a preserved space-time supersymmetry is a certain linear combination of a left- and right-moving charge that annihilates the boundary state [41, 42]:

$$(Q_L + Q'_R)\, |a\rangle_A^{(\mathrm{CY})} = 0\,. \tag{3.18}$$

If s is any left-moving spectral flow operator, $\omega_a(s)$ will again have the properties of a spectral flow, and therefore the sum of the corresponding left and right-moving zero modes will still constitute a conserved supersymmetry. That different boundary states in Gepner models conserve different space-time supersymmetry charges has been observed in [40]. What is new about the analysis above is the observation that this corresponds to dealing with boundary conditions of distinct automorphism types. Indeed, the fact that within the classification into A- and B-type there exist subclasses of boundary states with different automorphism type has so far not been appreciated in the literature.

Thus all boundary conditions in Gepner models that have been considered so far in the literature, and for which we have identified an automorphism type labeled by an element $\theta \in 2\pi\mathbb{Z}/N'_s$, in fact do preserve half of space-time supersymmetry. In short, in all situations studied in this paper, the presence of *space-time* supersymmetry in the closed string sector (ensured by the GSO projection) together with the preservation of $N=1$ *world sheet* supersymmetry by a boundary state already guarantees that the boundary state is BPS. On the other hand, the open string spectrum, encoded in the annulus partition function, will be space-time supersymmetric only if the automorphism types of the boundary conditions on the two boundary components of the annulus are equal. (Without reference to automorphism types, a condition of this type for space-time supersymmetry of the partition function was also derived in [37].) In that case all annuli can be expressed in terms of characters of the Calabi–Yau extension $\mathcal{C}^{(\mathrm{CY})}$, and as a consequence the GSO projection – which is a chiral issue – guarantees in particular the absence of tachyons in the spectrum of open string states. Geometrically, the difference $\theta_1 - \theta_2$ between the labels of the two automorphism types can be given the intuitive interpretation of an angle between two branes. The open string spectrum is space-time supersymmetric if the angle between the two branes is zero. In contrast, when the angle is non-zero, then absence of tachyons in the open string spectrum is not guaranteed any longer.

Thus in order to guarantee the absence of open string tachyons, among the A_θ-type boundary conditions it is generically necessary to restrict to those with a single fixed value of θ. For instance, when we decide to keep a boundary state with $\theta=0$, then we typically have to dismiss all boundary states with $\theta \neq 0$. However, in the case of type I theories, an orientifold projection may stabilize the brane (for reviews see [43,44]); in such circumstances, A_θ boundary conditions with several distinct values of θ could coexist. Whether this happens or not, and for what choices of θ, are model dependent questions. (The answer can in particular depend on bulk moduli.) Furthermore, when we also include B-type conditions, it can happen that requiring absence of open string tachyons restricts the allowed θ value of A_θ conditions even when only a single θ is kept.

3.4 A_0-type boundary conditions

The special class of boundary conditions of type $\mathrm{A}_0 \equiv \mathrm{A}_{\theta=0}$ are just those of 'trivial automorphism type' $\omega = \mathrm{id}$, i.e. those for which the identity map of $\mathfrak{A}^{(\mathrm{CY})}$ is used as the extension of the identity of $\mathfrak{A}_{\mathrm{int}}$. These boundary conditions preserve the whole algebra $\mathfrak{A}^{(\mathrm{CY})}$. Put differently, they are precisely the boundary conditions that were studied long ago by Cardy [13] for an arbitrary rational conformal field theory. In the case of tensor products of minimal models, various aspects of A_0-type conditions have already been studied in [37, 40, 48, 17]. As far as the space-time aspects of string theory are concerned, the A_0-type boundary conditions are not distinguished in any specific manner among the larger set of A-type boundary conditions, which as explained above preserve both world sheet and space-time supersymmetry, too. However, from a pure world sheet point of view, A_0-type boundary conditions are special in that they preserve the full chiral algebra $\mathfrak{A}^{(\mathrm{CY})}$ of the Calabi–Yau extension. This means that they are directly accessibly by Cardy's method, and we therefore still treat them separately here.

The only data that enter Cardy's construction of boundary states are the entries of the modular S-matrix of the conformal field theory, i.e. in our case the matrix (2.7) for the Calabi–Yau extension $\mathcal{C}^{(\mathrm{CY})}$. (In terms of the classifying algebra mentioned above, this result is a manifestation of the fact that the invariant subalgebra that corresponds to the A_0-type boundary conditions is nothing but the fusion rule algebra of $\mathcal{C}^{(\mathrm{CY})}$.) These data are well under control (some explicit formulas will be presented below). Besides the boundary states, also other aspects of these special boundary conditions are well understood (compare e.g. [45, 46] for a general discussion of correlation functions). Therefore the case of trivial automorphism type – which was also used as a starting point in the constructions in [15, 12] – is absolutely under control.

In particular, the A_0-type boundary states are in natural one-to-one correspondence with the primary fields of the Calabi–Yau extension $\mathcal{C}^{(\mathrm{CY})}$, and they can be expanded in the Ishibashi states of $\mathcal{C}^{(\mathrm{CY})}$:

$$|[\mu,\psi_\mu]\rangle^{(\mathrm{CY})}_{A_0} = \sum_{[\lambda,\psi_\lambda]} \frac{(S^{(\mathrm{CY})})_{[\lambda,\psi_\lambda],[\mu,\psi_\mu]}}{\sqrt{(S^{(\mathrm{CY})})_{[\Omega],[\mu,\psi_\mu]}}}\, |[\lambda,\psi_\lambda]\rangle\!\rangle \,. \tag{3.19}$$

Similarly, in the formula (3.8) for the annulus coefficients we now have $\mathcal{K}_{\omega_1\omega_2} = \mathcal{K}^{(\mathrm{CY})}$ as well as $\tilde{S} = S' = S^{(\mathrm{CY})}$, so that it reduces to

$$\mathrm{A}_{[\mu_1,\psi_1],[\mu_2,\psi_2]}^{[\nu,\psi_\nu]} = \sum_{[\lambda,\psi_\lambda]} (S^{(\mathrm{CY})})^*_{[\lambda,\psi_\lambda],[\mu_1,\psi_1]} (S^{(\mathrm{CY})})_{[\lambda,\psi_\lambda],[\mu_2,\psi_2]} (S^{(\mathrm{CY})})_{[\lambda,\psi_\lambda],[\nu,\psi_\nu]} / (S^{(\mathrm{CY})})_{[\lambda,\psi_\lambda],[\Omega]} \,. \tag{3.20}$$

By comparison with the Verlinde formula for $\mathcal{C}^{(\mathrm{CY})}$ we then learn that the annulus coefficients indeed coincide with the structure constants of the fusion algebra of $\mathcal{C}^{(\mathrm{CY})}$.

3.5 Tensor products of minimal models

Let us now specialize to the Gepner models proper, where the inner sector $\mathcal{C}_{\rm int}$ is a tensor product $\mathcal{C}_{k_1k_2\dots k_r} = \mathcal{C}_{k_1} \otimes \cdots \otimes \mathcal{C}_{k_r}$ of $N=2$ minimal models $\mathcal{C}_{k_i}$ at levels $k_i \in \mathbb{Z}_{>0}$. We also restrict our attention to the A_0-type boundary conditions. In this special case it is particularly easy to make the formula for $\tilde{S}$, and hence the description of boundary states, fully explicit. We first recall the following information about such Gepner models that will be needed in the sequel.

- It ist convenient to think of an $N=2$ minimal model at level k as a coset construction $\mathfrak{su}(2)_k \times \mathfrak{u}(1)_4/\mathfrak{u}(1)_{2h}$, where $h := k+2$. Accordingly, we denote its primary fields by $\Phi_q^{l,s}$. Then the labels l, s, q are in the ranges $l \in \{0, 1, \dots, k\}$, $s \in \{0, 1, 2, 3\}$, and $q \in \{0, 1, \dots, 2h-1\}$, subject to the parity selection rule $l + s - q \in 2\mathbb{Z}$, and to the 'field identification' [47, 3] $\Phi_q^{l,s} \equiv \Phi_{q+h}^{k-l,s+2}$. This labeling of primary fields refers to the bosonic subalgebra of the $N=2$ superconformal algebra. For example, the two world sheet supercurrents are not regarded as descendants of the vacuum but, rather, both correspond to the primary field $\Phi_0^{0,2} \equiv \Phi_{\pm h}^{k,0}$. The identity primary field is $\Omega = \Phi_0^{0,0}$. (For more details about minimal models see, for example, [3].)
- The primary fields of $\mathcal{C}_{k_1k_2\dots k_r}$ are then labeled as

$$(\Phi_{q_1}^{l_1,s_1}, \Phi_{q_2}^{l_2,s_2}, \dots, \Phi_{q_r}^{l_r,s_r}) \tag{3.21}$$

where $\Phi_{q_i}^{l_i,s_i}$ is a primary field of the ith minimal model. For brevity, below we will also use the notation

$$(\lambda, \sigma, \xi) \equiv (l_1, s_1, q_1, \dots, l_r, s_r, q_r) \tag{3.22}$$

for these collections of labels.
- The simple current group $\mathcal{K}^{(\rm CY)}$ of the Calabi–Yau extension is of the form (2.18), with r the number of minimal model factors. It is generated by the $r-1$ order-two currents

$$\mathrm{w}_i := (v, \Omega, \dots, \Omega, v, \Omega, \dots, \Omega)\,, \qquad i \in \{2, 3, \dots, r\}\,, \tag{3.23}$$

where the v-entries are in the first (say) and ith minimal model, together with the combination

$$\mathrm{u} := (s, s, \dots, s)^2\, (v, \Omega, \dots, \Omega)^{d/2} = (s^2 v^{d/2}, s^2, \dots, s^2)\,. \tag{3.24}$$

Here v stands for the minimal model primary field $\Phi_0^{0,2}$ that contains the two world sheet supercurrents, and $s = \Phi_1^{0,1}$ is the simple current in the Ramond sector whose action provides the spectral flow. These fields

$$s = \Phi_1^{0,1} \quad \text{and} \quad v = \Phi_0^{0,2} \tag{3.25}$$

have conformal weight $c/24 = k/8(k+2)$ and $3/2$, respectively.
- The order of u is

$$\mathrm{ord}(\mathrm{u}) = N_s/2 = \mathrm{scm}_{i=1,2\dots r}\{\eta_i\, h_i\} \tag{3.26}$$

with $h_i \equiv k_i+2$ and $\eta_i = 1$ for k_i even, $\eta_i = 2$ for k_i odd.

Exploiting our knowledge about the minimal model fusion rules, it is straightforward combinatorics to establish the following group and fixed point structure of $\mathcal{K}^{(\rm CY)}$.
- When all levels k_i are odd, then we have

$$\mathrm{u}^{N_s/4} = \prod_{i=1}^{r} \mathrm{w}_i\,, \tag{3.27}$$

and hence $\mathcal{K}^{(\mathrm{CY})} = \mathbb{Z}_{N_s/2} \times \mathbb{Z}_2^{r-2}$. In this case there are no fixed points.

▪ In contrast, when at least one level is even, then we have $\mathcal{K}^{(\mathrm{CY})} = \mathbb{Z}_{N_s/2} \times \mathbb{Z}_2^{r-1}$, and fixed points do occur. In fact there is then a unique simple current $\mathrm{L} \in \mathcal{K}^{(\mathrm{CY})}$ having fixed points. L is given by

$$\mathrm{L} = \mathrm{u}^{N_s/4} \prod_{i=1}^{r} (\mathrm{w}_i)^{\epsilon_i} \,, \tag{3.28}$$

where the value of $\epsilon_i \in \{0,1\}$ depends on the power of 2 contained in $N_s/2$. Namely, according to (3.26), $N_s/2$ is always even; when it is also divisible by 4, then

$$\epsilon_i = \begin{cases} 1 & \text{when the power of 2 in } h_i \text{ is maximal}\,, \\ 0 & \text{else}\,. \end{cases} \tag{3.29}$$

When $N_s/2$ is not divisible by 4, then we have instead

$$\epsilon_i = \begin{cases} 1 & \text{when } h_i \text{ is odd}\,, \\ 0 & \text{else}\,. \end{cases} \tag{3.30}$$

It also follows that L has order 2, and that with the help of field identification it can be rewritten as

$$\mathrm{L} = (\Phi_0^{0,0}, \ldots, \Phi_0^{0,0}, \Phi_0^{k_{r'+1},0}, \ldots, \Phi_0^{k_r,0}) \,. \tag{3.31}$$

Here without loss of generality we assume that the h_i have been ordered in such a way that those containing the maximal power of 2 are the last $r{-}r'$ ones, i.e. are labeled by $i \in \{r'{+}1, \ldots, r\}$. Fixed points of L are all fields (3.21) which obey $l_i = k_i/2$ for every $i = r'{+}1, \ldots, r$, while all other labels are arbitrary (except, of course, for the parity selection rule $l_i{+}s_i{-}q_i \in 2\mathbb{Z}$ and the restriction to zero monodromy charge with respect to $\mathcal{K}^{(\mathrm{CY})}$).

Employing the general formula (2.7), we are thus in a position to display the modular S-matrix of the Calabi–Yau extension. For the tensor product theory $\mathcal{C}_{k_1 k_2 \ldots k_r}$ we have the tensor product of the S-matrices of the individual factors:

$$S_{(\lambda,\sigma,\xi),(\lambda',\sigma',\xi')} = 2^r \prod_{i=1}^{r} S^{\mathfrak{su}(2)_{k_i}}_{l_i,l'_i} \, S^{\mathfrak{u}(1)_4}_{s_i,s'_i} \, \big(S^{\mathfrak{u}(1)_{2h_i}}_{q_i,q'_i}\big)^* \tag{3.32}$$

(the r factors of 2 stem from field identification in each of the r minimal models). The fixed point matrix S^{L} reads

$$S^{\mathrm{L}}_{(\lambda,\sigma,\xi)^{\mathrm{L}},(\lambda',\sigma',\xi')^{\mathrm{L}}} = 2^r \prod_{i=1}^{r} S^{\mathfrak{u}(1)_4}_{s_i,s'_i} \, (S^{\mathfrak{u}(1)_{2h_i}}_{q_i,q'_i})^* \prod_{i=1}^{r'} S^{\mathfrak{su}(2)_{k_i}}_{l_i,l'_i} \prod_{i=r'+1}^{r} S^{\mathrm{J}\,\mathfrak{su}(2)_{k_i}}_{k_i/2,k_i/2} \,. \tag{3.33}$$

(This is only defined when both fields involved are fixed points, which is indicated by attaching the superscript of the labels $(\cdots)^{\mathrm{L}}$.) Here $S^{\mathrm{J}\,\mathfrak{su}(2)_k}$ is the fixed point ‘matrix’ for the simple current $l = k$ of the $\mathfrak{su}(2)$ WZW model at level k. As k acts by fusion as $k \star l = k{-}l$ and hence has just a single fixed point $k/2$, the quantity $S^{\mathrm{J}\,\mathfrak{su}(2)_k}$ is in fact a single number, and in this simple situation the results of [10] tell us immediately that $S^{\mathrm{J}\,\mathfrak{su}(2)_k}_{k/2,k/2} = \mathrm{e}^{-2\pi\mathrm{i}\cdot 3k/16}$. Moreover, $\sum_{i=r'+1}^{r} k_i/16$ must be a multiple of 1/12 [10], so that we have $\prod_{i=r'+1}^{r} S^{\mathrm{J}\,\mathfrak{su}(2)_{k_i}}_{k_i/2,k_i/2} = \mathrm{e}^{\pi\mathrm{i}\,p/2}$ with

$p\in\{0,1,2,3\}$ (inspecting the list of Gepner models, one sees that all values of p occur). For further simplification, notice that

$$\prod_{i=r'+1}^{r} (S^{\mathrm{J}\,\mathfrak{su}(2)_{k_i}})^2 = (-1)^{\sum_{i=r'+1}^{r} 3k_i/4} = (-1)^{\sum_{i=r'+1}^{r} k_i/4} \tag{3.34}$$

and that $\sum_{i=r'+1}^{r} k_i/4$ is always an integer, as follows from the condition on the central charge.

Putting this information together, we see that the modular S-matrix of $\mathcal{C}^{(\mathrm{CY})}$ reads

$$\begin{aligned} S^{(\mathrm{CY})}_{[(\lambda,\sigma,\xi),\psi],[(\lambda',\sigma',\xi'),\psi']} &= 2^{2r-2-\eta} N_s \prod_{i=1}^{r} S^{\mathfrak{u}(1)_4}_{s_i,s'_i}\, (S^{\mathfrak{u}(1)_{2h_i}}_{q_i,q'_i})^* \prod_{i=1}^{r'} S^{\mathfrak{su}(2)_{k_i}}_{l_i,l'_i} \\ &\qquad \Big[\big(1-\tfrac34 \prod_{i=r'+1}^{r} \delta_{l_i,k_i/2}\delta_{l'_i,k_i/2}\big) \prod_{i=r'+1}^{r} S^{\mathfrak{su}(2)_{k_i}}_{l_i,l'_i} + \tfrac14\,\psi\psi' \prod_{i=r'+1}^{r} \delta_{l_i,k_i/2}\delta_{l'_i,k_i/2}\, S^{\mathrm{J}\,\mathfrak{su}(2)_{k_i}}_{k_i/2,k_i/2}\Big] \\ &= \frac{2^{r-2-\eta}\,N_s}{\prod_{i=1}^{r} h_i} \prod_{i=1}^{r} \mathrm{e}^{\pi\mathrm{i}(q_iq'_i/h_i - s_is'_i/2)} \prod_{i=1}^{r'} \sin\big(\frac{(l_i{+}1)(l'_i{+}1)\pi}{h_i}\big) \\ &\qquad \Big[\big(1-\tfrac34 \prod_{i=r'+1}^{r} \delta_{l_i,k_i/2}\delta_{l'_i,k_i/2}\big) \prod_{i=r'+1}^{r} \sin\big(\frac{(l_i{+}1)(l'_i{+}1)\pi}{h_i}\big) \\ &\qquad\qquad + 2^{-2+(r-r')/2}\, (-1)^{\sum_{i=r'+1}^{r} k_i/4}\, \psi\psi' \prod_{i=r'+1}^{r} \delta_{l_i,k_i/2}\delta_{l'_i,k_i/2}\, h_i^{1/2}\Big]\,. \end{aligned} \tag{3.35}$$

Recall that $\eta=1$ when all levels are odd, in which case there are no fixed points, and $\eta=0$ else. Upon insertion of (3.35) into the Verlinde formula, one obtains the fusion rule coefficients of $\mathcal{C}^{(\mathrm{CY})}$; for $\eta=0$ we arrive at the following expression (for $\eta=1$ the formula is similar, but without the complications involving fixed points):

$$\begin{aligned} &(\mathrm{N}^{(\mathrm{CY})})_{[(\lambda,\sigma,\xi),\psi],[(\lambda',\sigma',\xi'),\psi']}^{\qquad [(\lambda'',\sigma'',\xi''),\psi'']} = \frac{1}{|\mathcal{S}_\lambda|\,|\mathcal{S}_{\lambda'}|\,|\mathcal{S}_{\lambda''}|} \sum_{n=0,\dots,N_s/4-1}\ \sum_{\substack{\epsilon_j=0,1\ \forall\ j=1,2\dots,r \\ \sum_j \epsilon_j=0 \bmod 2}} \\ &\prod_{i=1}^{r'} \big(\delta_{\Delta s_i+2n+2\epsilon_i}\delta_{\Delta q_i+2n}\, \mathrm{N}_{l_i,l'_i}^{\ \ l''_i} + \delta_{\Delta s_i+2n+2\epsilon_i+2}\delta_{\Delta q_i+2n+h_i}\, \mathrm{N}_{l_i,k_i-l'_i}^{\ \ l''_i}\big) \\ &\ \Big\{ \textstyle\prod_{i=r'+1}^{r} \big(\delta_{\Delta s_i+2n+2\epsilon_i}\delta_{\Delta q_i+2n}\, \mathrm{N}_{l_i,l'_i}^{\ \ l''_i} + \delta_{\Delta s_i+2n+2\epsilon_i+2}\delta_{\Delta q_i+2n+h_i}\, \mathrm{N}_{l_i,k_i-l'_i}^{\ \ l''_i}\big) \\ &\quad + \textstyle\prod_{i=r'+1}^{r} \big(\delta_{\Delta s_i+2n+2\epsilon_i}\delta_{\Delta q_i+2n}\, \mathrm{N}_{l_i,k_i-l'_i}^{\ \ l''_i} + \delta_{\Delta s_i+2n+2\epsilon_i+2}\delta_{\Delta q_i+2n+h_i}\, \mathrm{N}_{l_i,l'_i}^{\ \ l''_i}\big) \\ &\quad + 2\, \textstyle\prod_{i=r'+1}^{r} \delta_{\Delta s_i+2n+2\epsilon_i}\delta_{\Delta q_i+2n} \Big[\psi\psi' \prod_{i=r'+1}^{r} (S^{\mathrm{J}\,\mathfrak{su}(2)_{k_i}})^2\, \delta_{l_i,k_i/2}\delta_{l'_i,k_i/2} \dfrac{S^*_{l''_i,k_i/2}}{S_{0,k_i/2}} \\ &\qquad\quad + \psi'\psi'' \textstyle\prod_{i=r'+1}^{r} \delta_{l'_i,k_i/2}\delta_{l''_i,k_i/2} \dfrac{S_{l_i,k_i/2}}{S_{0,k_i/2}} + \psi\psi'' \prod_{i=r'+1}^{r} \delta_{l_i,k_i/2}\delta_{l''_i,k_i/2} \dfrac{S_{l'_i,k_i/2}}{S_{0,k_i/2}}\Big] \\ &\quad + 2\, \textstyle\prod_{i=r'+1}^{r} \delta_{\Delta s_i+2n+2\epsilon_i+2}\delta_{\Delta q_i+2n+h_i} \Big[\psi\psi' \prod_{i=r'+1}^{r} (S^{\mathrm{J}\,\mathfrak{su}(2)_{k_i}})^2\, \delta_{l_i,k_i/2}\delta_{l'_i,k_i/2} \dfrac{S^*_{k_i-l''_i,k_i/2}}{S_{0,k_i/2}} \\ &\qquad\quad + \psi'\psi'' \textstyle\prod_{i=r'+1}^{r} \delta_{l'_i,k_i/2}\delta_{l''_i,k_i/2} \dfrac{S_{k_i-l_i,k_i/2}}{S_{0,k_i/2}} + \psi\psi'' \prod_{i=r'+1}^{r} \delta_{l_i,k_i/2}\delta_{l''_i,k_i/2} \dfrac{S_{k_i-l'_i,k_i/2}}{S_{0,k_i/2}}\Big]\Big\}\,. \end{aligned} \tag{3.36}$$

Concerning the notation, the following remarks are in order. First, we put

$$\Delta s_i := s_i + s_i' - s_i'' \,, \qquad \Delta q_i := q_i + q_i' - q_i'' \,. \tag{3.37}$$

Second, the factors $\delta_x \equiv \delta_{x,0}$ represent the fusion coefficients of the various $\mathfrak{u}(1)$ factors; in particular, an appropriate periodicity in their subscripts is understood. Third, we have separated the $\mathcal{K}^{\rm (CY)}$-summation into a part involving the simple currents without fixed points ($\sum_{n,\epsilon_i}$) and one that implements the order-two simple current L (the expression in curly brackets), compare formula (2.10). Further, the innermost (pairwise) summation takes into account the field identification in the minimal models. Finally, $\psi \equiv \psi(\mathrm{L}) \in \{\pm 1\}$ corresponds to the two irreducible characters of the group $\mathbb{Z}_2 = \{\Omega, \mathrm{L}\}$.

The terms in formula (3.36) that involve factors of ψ can be simplified by using the identity (3.34) and noting that

$$\frac{S_{l,k/2}}{S_{0,k/2}} = \sin\big(\tfrac{\pi}{2}\,(l+1)\big) = \left\{ \begin{array}{rl} (-1)^{l/2} & \text{if } l \text{ is even}\,, \\ 0 & \text{if } l \text{ is odd}\,. \end{array} \right. \tag{3.38}$$

Both of the expressions in square brackets are thus non-zero only if the labels l_i, l_i', l_i'' are even for all $i \in \{r'{+}1, r'{+}2, \ldots, r\}$, in which case they read

$$\psi\psi'\,(-1)^{\sum_{i=r'+1}^{r} k_i/4+l_i''/2}\,\Pi\,\Pi' + \psi'\psi''\,(-1)^{\sum_{i=r'+1}^{r} l_i/2}\,\Pi'\,\Pi'' + \psi\psi''\,(-1)^{\sum_{i=r'+1}^{r} l_i'/2}\,\Pi\,\Pi''\,, \tag{3.39}$$

where we introduced $\Pi := \prod_{i=r'+1}^{r} \delta_{l_i,k_i/2}$ and analogously Π' and Π''. We also mention that the annulus coefficients for annuli with two boundary conditions of A_0-type are just given by the fusion rules, so that the expression (3.36) directly provides us with the multiplicities of the corresponding open string states.

3.6 A_0-type boundary conditions for minimal model tensor products

Having collected these results about minimal model tensor products, we are now in a position to write down the A_0-type boundary states for the corresponding Calabi–Yau extensions $\mathcal{C}^{\rm (CY)}$. According to Cardy's results, the labeling of the A_0-type boundary conditions is *precisely the same* as for the primary fields of $\mathcal{C}^{\rm (CY)}$. Let us make this explicit. We start with the collection $(\lambda,\sigma,\xi) \equiv (l_1,s_1,q_1,\ldots,l_r,s_r,q_r)$ of labels ranging over $l_i \in \{0,1,\ldots,k_i\}$, $s_i \in \{0,1,2,3\}$, and $q_i \in \{0,1,\ldots,2h_i\}$. We then implement the various projections by imposing the following selections and identifications (both on bulk fields and on boundary conditions).

- Selections:

We impose $l_i + s_i + q_i \in 2\mathbb{Z}$ (minimal model selection rule), $s_1 + s_i \in 2\mathbb{Z}$ for all $i = 2, 3, \ldots, r$ (fermion alignment), and $Q \equiv \sum_{i=1}^{r}(-q_i/h_i + s_i/2) - s_1 d/4 \in \mathbb{Z}$ (charge projection).

- Identifications:

We restrict the l_i to the range $0 \le l_i \le k_i/2$ (minimal model field identification). Representatives for the orbits with respect to the alignment currents w_i can be labeled by a single $s \in \{0,1\}$ when $d/2 + r$ is odd, and $s \in \{0,1,2,3\}$ when $d/2 + r$ is even; all the s_i are equal to $s \bmod 2$.
Implementation of the identification implied by the current u is more difficult; it involves the divisibility properties of the heights h_i, which in general do not have a simple structure. In special cases, for instance when $\mathrm{lcm}_i\{h_i\} = h_j$ for some j, the corresponding label q_j can be set to zero using this identification.

Explicit formulas for boundary states of Gepner models were first presented in [37], where the boundary states were expressed in terms of the modular S-matrices of the $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ building blocks of the minimal models. However, as explained above, the chiral algebra $\mathfrak{A}^{(\mathrm{CY})}$ of the Gepner model is much larger than the chiral algebra $\mathfrak{A}_{k_1 k_2 \ldots k_r}$ of the tensor product $\mathcal{C}_{k_1 k_2 \ldots k_r}$ of minimal models. Accordingly, in the bulk the modular transformations are described by the extended S-matrix $S^{(\mathrm{CY})}$. The non-trivial information contained in $S^{(\mathrm{CY})}$ that cannot be obtained from the tensor product S-matrix alone stems from the presence of fixed points under the Calabi–Yau extension. Once $S^{(\mathrm{CY})}$ has been determined, the usual description of boundary conditions that preserve the full chiral algebra [13, 49] can be applied, though it is now to be formulated with the help of the matrix $S^{(\mathrm{CY})}$ rather than S. It is therefore not guaranteed that the boundary states can be written in the form presented in [37]. That the modular S-matrix must be 'properly resolved' for the construction of boundary states in Gepner models was emphasized in [48]. It was also noticed in [48] that some of the boundary states given in [37] are not consistent with all projections in Gepner models, which explains certain discrepancies between the results of [40] and [48]. More recently, it has been established in [17] that some of the boundary states of [37] are not elementary.

In [17] A-type boundary states for Gepner models with $D=4$ and $r=5$ were constructed by implementing the simple current extension (including in particular fixed point resolution) directly on the boundary states of the underlying $N=2$ tensor product. On the other hand, the results reported above clearly also allow to obtain these A-type boundary states by performing the extension already in the *bulk*. When doing so, one obtains the boundary states by merely combining standard results for simple current extensions in the bulk (which lead in particular to the formula (3.35) for the modular S-matrix $S^{(\mathrm{CY})}$ of the Calabi–Yau extension) with the general results of Cardy [13] for symmetry preserving boundary conditions in arbitrary rational conformal field theories. Let us point out that this way we get the A-type boundary states for *every* Gepner model, i.e. for $D=4,6,8$ and for any allowed r. [10] Moreover, when proceeding in this manner the boundary states are completely determined, up to over-all normalization, by the algorithm. Thus there is e.g. no need to invoke integrality of the annulus coefficients in order to fix the relative strength of the contributions from twisted and untwisted sectors. Indeed, integrality of the annulus coefficients involving only A_0-type boundary conditions coefficients is just the Verlinde formula as applied to the extended theory, i.e. to the Calabi–Yau extension.

More concretely, the A_0-type boundary states are given by the expression (3.19),

$$|[(\lambda,\sigma,\xi),\psi]\rangle_{A_0}^{(\mathrm{CY})} = \sum_{[(\lambda',\sigma',\xi'),\psi']} \frac{\big(S^{(\mathrm{CY})}\big)_{[(\lambda,\sigma,\xi),\psi],[(\lambda',\sigma',\xi'),\psi']}}{\sqrt{\big(S^{(\mathrm{CY})}\big)_{[0,0,0],[(\lambda',\sigma',\xi'),\psi']}}}\, |[(\lambda',\sigma',\xi'),\psi']\rangle\!\rangle \tag{3.40}$$

with $S^{(\mathrm{CY})}$ as presented in formula (3.35) and with the summation ranging over all primaries of $\mathcal{C}^{(\mathrm{CY})}$, [11] and the annulus coefficients for two boundary conditions of A_0-type coincide with the fusion rules (3.36). In the special case that (λ,σ,ξ) is *not* a fixed point of the current L (3.28),

[10] To make contact to geometry, $D/2+r$ must be odd. This can, however, always be achieved by introducing a trivial $k=0$ minimal model as an additional factor of the tensor product.

[11] We may wish to split the boundary state $|a\rangle_A^{(\mathrm{CY})}$ into its contributions involving only the Ishibashi states for full $\mathcal{K}^{(\mathrm{CY})}$-orbits and only those for fixed points, respectively. Then the number $\prod_{i=r'+1}^{r} h_i^{1/2}$ that is built in $S^{(\mathrm{CY})}$ (see the last line of formula (3.35)) appears as a factor between the two parts. In [17] this factor was obtained by imposing integrality of annulus coefficients.

the expression (3.40) for the boundary state reduces to

$$\begin{aligned} |[(\lambda,\sigma,\xi)]\rangle_{A_0}^{(\mathrm{CY})} &= \Big[\frac{2^{r-2-\eta}N_s}{\prod_{j=1}^r h_j}\Big]^{1/2} \sum_{[(\lambda',\sigma',\xi'),\psi']} \\ &\qquad \prod_{i=1}^{r} \mathrm{e}^{\pi\mathrm{i}(q_iq_i'/h_i - s_is_i'/2)}\, \frac{\sin\big((l_i{+}1)(l_i'{+}1)\pi/h_i\big)}{\big(\sin\big((l_i'{+}1)\pi/h_i\big)\big)^{1/2}}\, |[(\lambda',\sigma',\xi'),\psi']\rangle\!\rangle\,. \end{aligned} \tag{3.41}$$

For the boundary state in the full Gepner model, one has to combine this expression with the space-time part of the boundary state, include a phase factor $S^{\mathrm{D}_{D/2+3}}_{\varpi,\varpi'}/\sqrt{S^{\mathrm{D}_{D/2+3}}_{o,\varpi'}}$, a sign from undoing the bosonic string map, and a factor of 2 from the remaining projections. The resulting formula is then essentially the one reported in [37]; but even in this special case our result differs in the power of 2 that appears in the prefactor.

In contrast, when (λ,σ,ξ) *is* a fixed point of L, then the additional terms in the formula (3.35) for $S^{(\mathrm{CY})}$ contribute, and the expression for the boundary state gets a bit lengthier. Boundary states $|[(\lambda,\sigma,\xi),\psi]\rangle_{A_0}^{(\mathrm{CY})}$ corresponding to resolved fixed points have been studied in [17]. Apart from the proper power of 2, our result differs from the one presented in [17] also by the absence of a factor of $\sqrt{N_s}$ in the ψ-dependent terms.

3.7 General A-type boundary conditions

Boundary conditions of automorphism type A_θ with $\theta\neq 0$ cannot be obtained from Cardy's results alone. In this subsection, we explain more explicitly how the generalizations developed in [15, 12] allow to structurize and construct all A-type boundary conditions. We are faced with the following situation. Given the theory $\mathcal{C}^{(\mathrm{CY})}$, obtained from $\mathcal{C}_{\mathrm{int}}$ by a simple current extension with $\mathcal{K}^{(\mathrm{CY})}$, we want to construct all boundary conditions that preserve at least the chiral symmetry algebra of $\mathcal{C}_{\mathrm{wsusy}}$. Here $\mathcal{C}_{\mathrm{wsusy}}$ is the theory obtained from $\mathcal{C}_{\mathrm{int}}$ by extension with the alignment currents w_i alone; this means in particular that all boundary conditions preserve world sheet supersymmetry.

The solution to this problem is as follows. The set of all such boundary conditions is the set of irreducible representations of the classifying algebra $\mathfrak{C}\equiv\mathfrak{C}(\mathcal{C}^{(\mathrm{CY})}{:}\,\mathcal{C}_{\mathrm{wsusy}})$. This set of boundary conditions can be divided into subsets of definite automorphism type, here labeled by the angle 2θ with $\theta/2\pi\in\mathbb{Z}/N_s'$. (Recall from subsection 3.3 that θ can be interpreted as specifying which space-time supersymmetry is preserved by a boundary condition.) Furthermore, each of these subsets furnishes the set of irreducible representations of an *individual classifying algebra* $\mathfrak{C}_{2\theta}$ which, just like $\mathfrak{C}$, is a semisimple commutative associative algebra over $\mathbb{C}$, and one has

$$\mathfrak{C} = \bigoplus_{\theta} \mathfrak{C}_{2\theta} \tag{3.42}$$

as a direct sum of algebras over $\mathbb{C}$. We remark that the same situation arises in much simpler models in statistical mechanics, too. In the critical three-state Potts model, for instance, there are eight boundary conditions which come in two automorphism types that are distinguished by the reflection condition for the W_3-current [50]. Accordingly, in that case the classifying algebra is a direct sum $\mathfrak{C}_+\oplus\mathfrak{C}_-$ of a six- and a two-dimensional algebra; the irreducible representations of $\mathfrak{C}_+$ provide the three fixed and the three mixed boundary conditions of the Potts model, while the ones of $\mathfrak{C}_-$ provide the free and 'new' [51] conditions.

In general there is no simple relationship between the various individual classifying algebras $\mathfrak{C}_{2\theta}$. In the case of Gepner models, however, it turns out that there exist symmetries generated by simple currents of the theory $\mathcal{C}_{\rm wsusy}$ (the so-called 'phase symmetries'), which act on the set of boundary conditions and thereby relate individual classifying algebras for different θ with each other. Because of those symmetries, the boundary conditions for $\theta \neq 0$ look still very similar to the Cardy case. It follows in particular that the combinatorics of fixed points and their resolution do not depend on the value of θ. (This can already be deduced from formula (3.31) for the fixed point current L.) Those symmetries between boundary conditions with different θ have been implicitly used in [52] for a nice organization of A-type boundary conditions. Note, however, that this simplification is intrinsically linked to the special symmetries of $N=2$ minimal models and cannot be expected to be present in general.

For a more detailed description, we recall that the group that furnishes the extension from $\mathcal{C}_{\rm wsusy}$ to $\mathcal{C}^{\rm (CY)}$ is generated by the image U of the simple current u (3.24) in the $\mathcal{C}_{\rm wsusy}$-theory, and that this current has order $N_s'/2$ with N_s' as defined in formula (2.20), i.e. $N_s' = N_s/2^\eta$ with $\eta \in \{0,1\}$. The primary fields of $\mathcal{C}_{\rm wsusy}$ are labeled by $\Lambda := [(\lambda,\sigma,\xi)]'$ where the prime indicates that orbits are taken with respect to the group generated by the alignment currents only (which do not have fixed points); thus in particular $\mathrm{U} = [\mathrm{u}]'$. A distinguished basis $\{\tilde\Phi_\Lambda\}$ of the classifying algebra $\mathfrak{C}$ is then labeled by the set of all $\mathcal{C}_{\rm wsusy}$-fields Λ that have vanishing monodromy charge with respect to U, together with a $\mathbb{Z}_2$ character accounting for a possibly non-trivial stabilizer. Further, natural bases of the *individual* classifying algebras $\mathfrak{C}_{2\theta}$ are provided by twisted sums $\tilde\Phi^\theta_\Lambda := \sum_{j=0}^{(N_s'/2)-1} \mathrm{e}^{2\mathrm{i}j\,\theta} \tilde\Phi_{\mathrm{U}^j\Lambda}$ of the basis elements $\tilde\Phi_\Lambda$ of $\mathfrak{C}$. The set of all boundary conditions, on the other hand, is labeled by the set of *orbits* $[\Lambda,\psi]''$ of fields of $\mathcal{C}_{\rm wsusy}$ with respect to the extension by U (again with proper account for stabilizers), but without restriction on the value of $Q_{\rm U}$. The set of boundary conditions with fixed automorphism type A_θ then corresponds to orbits $[\Lambda,\psi]''$, with the same minimal model and fermion alignment selection and identification rules as for $\theta = 0$, but with different charge projection condition

$$Q_{\rm U}(\Lambda) = \theta/\pi \bmod \mathbb{Z}\,. \tag{3.43}$$

Hereby in particular the counting of all A_θ-type boundary states is reduced to simple, if lengthy, combinatorics, which can (and should) be directly implemented in a computer algorithm.

4 Remarks on B-type boundary conditions

So far we have restricted our attention exclusively to A-type boundary conditions. Recall that these need not preserve all of the chiral algebra $\mathfrak{A}^{\rm (CY)}$ of the Calabi–Yau extension, but at least its subalgebra $\mathfrak{A}_{\rm wsusy}$, which in turn contains $\mathfrak{A}_{\rm int}$, the chiral algebra of the unextended inner sector $\mathcal{C}_{\rm int}$ (see the chain (2.19) of embeddings). Now we rather want to study boundary conditions that possess a non-trivial glueing condition already for the subalgebra $\mathfrak{A}_{\rm int}$. To have a well-defined total supercurrent for the tensor product, and for compatibility with the fermion alignment, we must use the same automorphism of the $N=2$ algebra for each factor of the tensor product $\mathcal{C}_{\rm int}$. As mentioned in subsection 3.1, generically the automorphism group we have to consider is the non-connected Lie group O(2). In the connected component of the identity of O(2) we can only use the identity element $\omega = \mathrm{id}$ itself; this gives rise to all A-type boundary conditions, of all subtypes discussed above. Here we are interested in automorphisms from the other connected component, which are characterized by

$$J_n \mapsto -J_n\,, \qquad G_r^+ \mapsto \mathrm{e}^{\mathrm{i}\gamma} G_r^-\,, \qquad G_r^- \mapsto \mathrm{e}^{-\mathrm{i}\gamma} G_r^+ \tag{4.1}$$

with $\gamma \in \mathbb{R}$. Unlike in the case of the identity component, where only the identity could be chosen, we can allow for any arbitrary value of γ. Boundary conditions obtained this way are referred to as B-*type* conditions. As a distinguished subset, they include those where the breaking is induced by the *mirror automorphism* of the total $N = 2$ algebra, which is obtained for $\gamma = 0$:

$$J_n \mapsto -J_n\,, \qquad G_r^\pm \mapsto G_r^\mp\,. \tag{4.2}$$

We will use the name B^C-*type* for these specific boundary conditions; the subscript C reminds of charge conjugation.

As compared to A-type conditions, we have to face a new problem, which also arises in other circumstances. Namely, we are only given an automorphism of the subalgebra $\mathfrak{A}_{\mathrm{int}}$, but not an automorphism of the full chiral algebra $\mathfrak{A}^{(\mathrm{CY})}$ of our interest. We have already seen above in the case $\omega = \mathrm{id}$ that there may exist several different lifts of an automorphism to a larger algebra. But at a more fundamental level, there is even no guarantee that any of the automorphisms of B-type can be lifted at all. Indeed, as explained in appendix A, in a general simple current extension there can exist an obstruction to lifting a given automorphism.

Fortunately, in the specific situation of interest to us here, there is in fact no such obstruction. This follows from the fact [38] that A- and B-type conditions get exchanged by the mirror map. In short, the B-type conditions of a Gepner model are well-defined for the full symmetry algebra $\mathfrak{A}^{(\mathrm{CY})}$ because they correspond to the A-type conditions of the mirror model, which by the results above are fully under control. In the bulk, the mirror map just amounts to applying charge conjugation in the inner sector. But at least in many Gepner models it can also be described alternatively [53] by forming the modular invariant that is associated [10] to a suitable group of simple currents of non-integral conformal weight. These currents implement the 'phase symmetries' of the minimal models. (In the literature [53–56] this is again usually referred to as an orbifold construction.) Accordingly, the methods of [33, 57, 58], which show how to deal with boundary conditions for torus partition functions associated to simple currents of half-integral conformal weight, should be helpful for analyzing the B-type conditions.

But still the concrete details in the description of the B-type conditions are rather involved. When formulated in terms of the chiral algebra $\mathfrak{A}_{\mathrm{int}}$, the complication manifests itself in the fact that the subalgebra that is preserved by the boundary conditions is neither contained in $\mathfrak{A}_{\mathrm{int}}$ nor does it contain $\mathfrak{A}_{\mathrm{int}}$. Roughly, one *simultaneously* tries to *reduce* the algebra – by taking the orbifold with respect to the automorphism group Γ in question, e.g. with respect to the $\mathbb{Z}_2$ generated by the mirror automorphism (4.2) – and to *extend* it – by the simple current extension with the group $\mathcal{K}^{(\mathrm{CY})}$. More concretely, we want to know a lift $\hat\Gamma$ of the orbifold group

Γ to $\mathfrak{A}^{(\mathrm{CY})}$ and a chiral algebra extension $\hat{\mathcal{K}}$ of $(\mathfrak{A}_{\mathrm{int}})^{\Gamma}$ to $(\mathfrak{A}^{(\mathrm{CY})})^{\hat{\Gamma}}$ such that the diagram

$$\begin{tikzcd} & \mathfrak{A}^{(\mathrm{CY})} \arrow[dr, "\hat{\Gamma}"] & \\ \mathfrak{A}_{\mathrm{int}} \arrow[ur, "\mathcal{K}^{(\mathrm{CY})}"] \arrow[rr, dashed] \arrow[dr, "\Gamma"'] & & (\mathfrak{A}^{(\mathrm{CY})})^{\hat{\Gamma}} \\ & (\mathfrak{A}_{\mathrm{int}})^{\Gamma} \arrow[ur, "\hat{\mathcal{K}}"'] & \end{tikzcd} \tag{4.3}$$

is commutative. Then in particular the map indicated by the dashed arrow is well defined. On the other hand, for our study of boundary conditions the crucial issue is the relation between the largest ($\mathfrak{A}^{(\mathrm{CY})}$) and smallest ($(\mathfrak{A}_{\mathrm{int}})^{\Gamma}$) chiral algebra in the diagram. The latter is the orbifold of the former by some automorphism group $\Gamma^{(\mathrm{CY})}$, and conversely, $\mathfrak{A}^{(\mathrm{CY})}$ is a certain extension $\mathcal{E}$ of $(\mathfrak{A}_{\mathrm{int}})^{\Gamma}$:

$$\begin{tikzcd} & \mathfrak{A}^{(\mathrm{CY})} \arrow[d, shift right, "\Gamma^{(\mathrm{CY})}"'] \\ \mathfrak{A}_{\mathrm{int}} \arrow[ur, "\mathcal{K}^{(\mathrm{CY})}"] \arrow[dr, "\Gamma"'] & \\ & (\mathfrak{A}_{\mathrm{int}})^{\Gamma} \arrow[u, shift right, "\mathcal{E}"'] \end{tikzcd} \tag{4.4}$$

Note that $\Gamma^{(\mathrm{CY})}$ is not the direct product $\Gamma\times\mathcal{G}$ of the orbifold group Γ (here $\mathbb{Z}_2$) and the simple current group $\mathcal{G}$ (here $\mathcal{K}^{(\mathrm{CY})}$). In fact, the current u (3.24) is not invariant under the mirror automorphism, so that even if one dealt with a group constructed out of Γ and $\mathcal{G}$, it could definitely not be their direct product. Closer inspection shows that the mirror orbifold $(\mathfrak{A}_{\mathrm{int}})^{\Gamma}$ of the tensor product theory is an orbifold of the Calabi–Yau extension $\mathfrak{A}^{(\mathrm{CY})}$ by a *non-abelian* group $\Gamma^{(\mathrm{CY})}$. Conversely, in order to obtain the orbifold of $\mathcal{C}^{(\mathrm{CY})}$ from the mirror orbifold of $\mathcal{C}_{\mathrm{int}}$ one must consider an extension $\mathcal{E}$ of the chiral algebra by fields not all of which are simple currents, though still all have integral quantum dimension. One may expect that $\hat{\Gamma}$ is a subgroup of $\Gamma^{(\mathrm{CY})}$, and that the fields in $\hat{\mathcal{K}}$ form a subset of those in $\mathcal{E}$. Some of the necessary mathematical tools for attacking this problem have recently been established (see e.g. [24, 59–62]), but they are not sufficient to obtain a complete description of the associated B-type boundary conditions.

However, we can still draw some general conclusions about B-type conditions by again invoking mirror symmetry. As a matter of fact, the statement that A- and B-type conditions get exchanged by the mirror symmetry has to be refined, because the mirror automorphism (4.2) only involves the total $N=2$ superconformal algebra of the tensor product, whereas as noticed above for the complete specification of A-type (and likewise of B-type) boundary conditions we have to prescribe the action of the automorphism on the full chiral algebra of the Calabi–Yau extension.[12] To do so, we use the fact that, as a special case of more general T-duality relations [12], those boundary conditions of a conformal field theory with torus partition

[12] In addition, one should be aware of the fact that initially the mirror map refers to the situation *before* applying the bosonic string map. However, the mirror map is compatible with the bosonic string map, so that we can directly apply it here. The point is that, in conformal field theory terminology, the mirror map is nothing but charge conjugation. But as a consequence of formula (2.2) we have $\tilde{C}B=BC$, where C is the charge

function Z which leave invariant precisely the subalgebra of the chiral algebra that is pointwise fixed under the charge conjugation automorphism ω^C can be put in one-to-one correspondence with the boundary conditions that preserve the full chiral algebra of a different conformal field theory, namely the one with torus partition function CZ.[13] It follows that the A_0-type boundary conditions of a Gepner model correspond to the B^C-type boundary conditions of its mirror model, and vice versa. How this generalizes to the more general A- and B-type conditions will be discussed elsewhere. In particular, this consideration tells us that there is no obstruction to a lift of the mirror automorphism of the total $N=2$ algebra to the full chiral algebra of the Calabi–Yau extension.

Finally let us mention that whenever the tensor product $\mathcal{C}_{\mathrm{int}}$ contains identical factors, there are additional automorphisms, beyond those belonging to O(2) discussed above, that can be modded out without spoiling world sheet supersymmetry, namely the permutation symmetries that interchange the identical factors.

5 Fixed points and singularities

We conclude this paper with a few comments on the relationship between fixed point structures in Gepner models and various other 'singular' structures that occur in the analysis of string compactifications on Calabi–Yau spaces and in Gepner models. The following list summarizes various such structures.

- It is generally believed that a Gepner model describes the exact solution of string theory compactified on a certain Calabi–Yau manifold, at a specific point of its moduli space. There is a general prescription for finding the polynomial constraints that provide the embedding of the Calabi–Yau manifold in a weighted projective space, see e.g. [63, 5, 64]. When carrying this construction through, one encounters the problem that the variety defined by those polynomial constraints is not smooth, but has singularities. It is only after resolving these singularities that one obtains the Calabi–Yau manifold.
- The moduli of the Calabi–Yau manifold are related to the (c, c) and (a, c) rings of the $N=2$ superconformal field theory [65]. For instance, deformations of the complex structure of the Calabi–Yau space correspond to Gepner model fields that are chiral primaries with respect to both the left- and right-moving chiral algebras, with total $\mathfrak{u}(1)$ charges 1. Fields that have in addition identical left- and right-moving labels in each minimal model can be related to polynomial deformations of the complex structure, i.e. to a change in the defining polynomial. But it has been pointed out long ago [66] that polynomial deformations give neither a complete nor an unambiguous enumeration of complex structure deformations. In the Gepner model, this can be related to the existence of twisted, i.e. left-right asymmetric, modes in the (c, c) ring, see e.g. [67].[14]
- In the study of boundary states in Gepner models and their comparison with geometric D-

conjugation matrix of the bosonic theory (i.e., after the bosonic string map B) and $\tilde{C}$ the charge conjugation matrix of the supersymmetric theory (before the bosonic string map).

[13] The simplest instance of this phenomenon is the exchange between Neumann and Dirichlet conditions in the theory of a single free boson.

[14] These states can be obtained directly by the algorithms incorporating the simple current extension, e.g. by the computer programs that were used to produce the bulk spectra in [3, 5, 67, 26, 28].

branes and bundles in the corresponding Calabi–Yau manifolds [52,68–70,7], it was noticed that several boundary states constructed in [37] are not elementary, in the sense that the annulus coefficient of the vacuum field is larger than one [68]. In [69] it has been argued that this can be understood if one assumes that the relevant boundary states correspond to branes carrying reducible bundles.

There have been speculations that (some of) these different singular structures are intimately related. Now in order for any definite relationship between geometrical and conformal field theory structures to exist, at least the combinatorial data characterizing them should be similar. In the situation at hand, this is actually not the case.

- Let us first recall some of our results concerning A-type boundary conditions. As we have explained, the construction of A-type boundary states in Gepner models is completely under control, and only existing technology [13, 21, 15, 12] is needed.[15] We have also seen that the fixed points in Gepner models are always of $\mathbb{Z}_2$ type, i.e. the only non-trivial stabilizer group that occurs is of the form $\{\Omega, \mathrm{L}\}$. On the other hand, singularities in the construction of the Calabi–Yau manifold as a complete intersection in a weighted projective space can occur if the weights have common divisors, and are locally of the type $\mathbb{C}^d/\mathbb{Z}_\ell$ where $d = 2, 3$ and ℓ can be any integer, see, for example, [71].
- It is also unlikely that there is any relationship between twisted modes in the (c, c) ring and singularities in the construction of the Calabi–Yau space. By inspecting lists of twisted (c, c) fields for particular Gepner models, it is easily checked that they do not display any $\mathbb{Z}_\ell$ structure. And indeed, the existence of non-polynomial deformations is not related to a singularity occuring in the construction of the manifold, but to the presence of an obstruction in the relevant cohomologies of the family of smooth manifolds [66]. On the conformal field theory side, twisted modes may also be related to the presence of an enhanced symmetry. For instance, compactification on K3 leads to $N = 4$ world sheet supersymmetry, and half of the modes in the (c, c) and (c, a) rings are twisted. Incidentally, in such a situation boundary conditions might break different parts of this extended symmetry, so that a finer classification of boundary conditions is possible.
- Furthermore, fixed points in the Gepner extension do not seem to be related to the presence of twisted modes in the (c, c) ring. In fact, the Gepner extension has fixed points whenever at least one level is even. But many of those Gepner models do not possess any twisted modes in their (c, c) ring. A simple example is provided by the Gepner model $(2, 2)$ which does have fixed points, but simply corresponds to a torus embedded in one-dimensional projective space, without any singularities.

Finally we would like to point out that in our opinion several interesting problems concerning boundary states in Gepner models and branes on Calabi–Yau compactifications are still open. For example, the construction of states charged under twisted modes in the (c, c) ring, as well as a better understanding of composite boundary states and their connection to reducible bundles, is highly desirable.

[15] In particular, the fixed points can already be understood entirely at the chiral level in closed string theory. It is therefore e.g. unnecessary to study open string partition functions in order to fix normalization factors (as advocated in [37, 17]). Rather, the annulus coefficients are non-negative integers by construction. Also, nothing is gained by separating boundary states or partition functions into a part from an 'untwisted' and one from a 'twisted' sector; all the relevant structures are already present in the closed string theory.

A Lifting orbifold automorphisms

In the main text we have seen that when analyzing B-type boundary conditions one must study the interplay between simple current extensions and orbifolds. Here we investigate this issue, which is of interest also in other situations, in its own right. Thus consider an arbitrary rational conformal field theory with chiral algebra $\mathfrak{A}$, a group $\mathcal{G}$ of simple currents (with integral conformal weight) of the theory, and a group Γ of automorphisms of $\mathfrak{A}$. Every automorphism ω of $\mathfrak{A}$ induces a permutation of the (labels for) primary fields, which is an automorphism of the fusion rules and which we denote by ω^*. By including the simple currents into the chiral algebra, one obtains an extended theory with chiral algebra $\mathfrak{A}_{\rm ext} \supset \mathfrak{A}$, while by dividing out the automorphisms in Γ one obtains an orbifold theory with chiral algebra $\mathfrak{A}^\Gamma \subset \mathfrak{A}$. Our goal is then to make sense of the symbol $\mathfrak{A}^\Gamma_{\rm ext}$. For simplicity and definiteness we restrict to the case $\Gamma = \mathbb{Z}_2$, i.e. there is only a single non-trivial automorphism ω and it has order two.

A necessary prerequisite for attaining this goal is to lift the automorphism ω to some automorphism $\omega_{\rm ext}$ of the extended chiral algebra $\mathfrak{A}_{\rm ext}$. As a matter of fact, our first task should be to determine whether such a lift is possible at all. Indeed there can be an obstruction, and this will be studied below. But for the moment let us restrict to those cases where the lifting of ω is not obstructed. In that case we need to investigate the uniqueness of the lift. We already know from the discussion in the main text that typically even the identity automorphism of $\mathfrak{A}$ will possess several distinct lifts to $\mathfrak{A}_{\rm ext}$. To be more concrete, we use the fact that we can characterize [15] $\mathfrak{A}$ as the fixed algebra of $\mathfrak{A}_{\rm ext}$ under an action of the group $G_{\rm ext} := \mathcal{G}^*$ of characters of the simple current group $\mathcal{G}$. Further, $\mathfrak{A}^\omega$ is the fixed algebra in $\mathfrak{A}$ under ω. Thus we can characterize $\mathfrak{A}^\omega$ equally well as the subalgebra of the big algebra $\mathfrak{A}_{\rm ext}$ under the combined action of $G_{\rm ext}$ and $\omega_{\rm ext}$, i.e. under the action of the group $H_{\rm ext}$ generated by $G_{\rm ext}$ and $\omega_{\rm ext}$. Note that we do *not* assume here that $\omega_{\rm ext}$ has order two.

We can then again employ the result from the Galois theory for vertex operator algebras that the possible chiral algebras between $\mathfrak{A}^\omega$ and $\mathfrak{A}_{\rm ext}$ are in one-to-one correspondence with subgroups of the group $H_{\rm ext}$. Now since $\mathfrak{A}$ is the fixed point subalgebra of $\mathfrak{A}_{\rm ext}$ under $G_{\rm ext}$, for every $g \in G_{\rm ext}$ the element $g_\omega := \omega_{\rm ext}^{-1} {\cdot} g {\cdot} \omega_{\rm ext} \in H_{\rm ext}$ acts on $\mathfrak{A}$ as $\omega^{-1} {\cdot} 1 {\cdot} \omega = 1$, and by the Galois correspondence every such g_ω already lies in $G_{\rm ext}$. This tells us that $G_{\rm ext}$ is a normal abelian subgroup of $H_{\rm ext}$, and the conjugation by $\omega_{\rm ext}$ acts on $G_{\rm ext}$ by an outer automorphism $\hat\omega$, i.e. $\omega_{\rm ext}^{-1}\, g\, \omega_{\rm ext} = \hat\omega(g)$. As $\omega_{\rm ext}^2 \in G_{\rm ext}$ and $G_{\rm ext}$ is abelian, the automorphism $\hat\omega$ has order two. So we have the structure [16]

$$0 \,\to\, G_{\rm ext} \,\to\, H_{\rm ext} \,\to\, \mathbb{Z}_2 \,\to\, 0\,. \tag{A.1}$$

As $\omega_{\rm ext}$ does not necessarily have order two, the term 'orbifold of $\mathfrak{A}_{\rm ext}$ by $\omega_{\rm ext}$' is to be interpreted as the orbifold by the cyclic subgroup of $H_{\rm ext}$ that is generated by $\omega_{\rm ext}$. But those elements of this subgroup that are of the form $(\omega_{\rm ext})^{2n}$ form a cyclic subgroup $G^0_{\rm ext}$ of $G_{\rm ext}$, so that we may as well perform the orbifolding stepwise, first by $G^0_{\rm ext}$ and afterwards by $\omega_{\rm ext}$ which then has order two on $(\mathfrak{A}_{\rm ext})^{G^0_{\rm ext}}$. It follows that at the price of possibly working with a different simple current extension than the original one, we may restrict our attention to the case when $\omega_{\rm ext}$ has order two.

[16] It is instructive to think of $H_{\rm ext}$ like of a compact Lie group with two connected components. The 'identity component' is $G_{\rm ext}$, and it has a natural unit element, while the other component, consisting of the automorphisms of $\mathfrak{A}_{\rm ext}$ whose restriction to $\mathfrak{A}$ acts like ω, does not possess a natural base point.

A simple illustration of the non-uniqueness of the lift is provided by the following $c=1$ theories. The original theory is the rational free boson X at compactification radius $R^2=mn^2$ with n integral and m even integral. Thus the chiral algebra $\mathfrak{A}$ is generated by the current $j=\mathrm{i}\partial X$, giving rise to $\mathfrak{u}(1)_{mn^2}$, and by the two fields $\Phi_\pm=\exp(\pm\mathrm{i}\sqrt{m}nX)$; there are mn^2 primary fields, which we label by the integers from 0 to $mn^2{-}1$ and each of which is a simple current. Extending this theory by the simple currents $\mathcal{G}=\{\ell mn\,|\,\ell=0,1,\ldots,n{-}1\}\cong\mathbb{Z}_n$, one obtains the theory of a free boson with compactification radius $R^2=m$ and chiral algebra $\mathfrak{A}_{\rm ext}$ generated by $\mathfrak{u}(1)_m$ and $\Phi^{\rm ext}_\pm=\exp(\pm\mathrm{i}\sqrt{m}X)$. On the other hand, dividing out the charge conjugation automorphism ω from $\mathfrak{A}$ one arrives at the $\mathbb{Z}_2$ orbifold of the free boson, with $mn^2/2+7$ primary fields. The map ω acts on the fields generating $\mathfrak{A}$ as

$$\omega(j) = -j\,, \qquad \omega(\Phi_\pm) = \Phi_\mp\,. \tag{A.2}$$

It can be lifted to $\mathfrak{A}_{\rm ext}$ as

$$\omega_{\rm ext}(j) = -j\,, \qquad \omega_{\rm ext}(\Phi^{\rm ext}_\pm) = \zeta^{\pm1}\,\Phi^{\rm ext}_\mp\,, \tag{A.3}$$

where ζ is an arbitrary nth root of unity. In terms of the free boson X, this reads

$$\omega_{\rm ext}(X) = -X + \frac{2\pi\ell}{n\sqrt{m}} \tag{A.4}$$

with some $\ell\in\{0,1,\ldots,n{-}1\}$. This example also displays nicely that even the identity map of $\mathfrak{A}$ can typically be lifted in several inequivalent ways. Clearly, for each nth root ζ of unity, the map

$$\mathrm{id}_{\rm ext}(j) = j\,, \qquad \mathrm{id}_{\rm ext}(\Phi^{\rm ext}_\pm) = \zeta^{\pm1}\,\Phi^{\rm ext}_\pm \tag{A.5}$$

(acting on the free boson field as $X\mapsto X+2\pi\ell/n\sqrt{m}$ for $\zeta=\exp(2\pi\mathrm{i}\ell/n)$) is an automorphism of $\mathfrak{A}_{\rm ext}$ and restricts to the identity map on $\mathfrak{A}$. Note that while the order of the map given by (A.3) is 2 independently of the value of ζ, this is no longer true for the map (A.5).

Let us now come back to the issue of existence of $\omega_{\rm ext}$. Under the most general circumstances such a lift may actually not exist at all. First, clearly the compatibility condition $\omega^*(\mathcal{G})=\mathcal{G}$ must be satisfied. In the sequel we assume that this is the case. (In the situation of our interest, this condition is indeed met. Also, the condition is automatically fulfilled whenever an extension is by *all* integer spin simple currents of a given theory.) But even with this assumption the existence of a lift $\omega_{\rm ext}$ is not guaranteed. Rather, one has to study the relation between the subgroup

$$\mathcal{G}_0 := \{\mathrm{J}\in\mathcal{G}\,|\,\omega^*(\mathrm{J})=\mathrm{J}\} \tag{A.6}$$

of $\mathcal{G}$ and the group $\mathcal{G}^\omega$ of simple currents of the $\mathfrak{A}^\omega$-theory. By general orbifold rules [11, 16], each $\mathrm{J}\in\mathcal{G}_0$ gives rise to two simple currents $\mathrm{J}_\pm$ in the untwisted sector of the orbifold theory $\mathfrak{A}^\omega$. The fields $\mathrm{J}_\pm$ form a subgroup $\mathcal{G}_0^\omega$ of $\mathcal{G}^\omega$. Thus $\mathcal{G}_0^\omega$ is a $\mathbb{Z}_2$-extension of $\mathcal{G}_0$, i.e. there is an exact sequence

$$0\,\rightarrow\,\mathbb{Z}_2\,\rightarrow\,\mathcal{G}_0^\omega\,\stackrel{\pi}{\rightarrow}\,\mathcal{G}_0\rightarrow 0\,, \tag{A.7}$$

where the projection π acts as $\pi(\mathrm{J}_+)=\mathrm{J}=\pi(\mathrm{J}_-)$. But $\mathcal{G}_0^\omega$ is not necessarily a direct product of $\mathcal{G}_0$ with $\mathbb{Z}_2$; the obstruction is expressed by an element $[\epsilon]$ in $H^2(\mathcal{G}_0,\mathbb{Z}_2)$.

Inspecting the fusion rules among the fields $\mathrm{J}_\pm$ in the orbifold theory $\mathfrak{A}^\omega$, one finds that this cohomology class $[\epsilon]$ has the following conformal field theory interpretation. The associated *commutator cocycle* $\bar\epsilon(\mathrm{J}_1,\mathrm{J}_2):=\epsilon(\mathrm{J}_1,\mathrm{J}_2)/\epsilon(\mathrm{J}_2,\mathrm{J}_1)$ (which only depends on the class $[\epsilon]$ and not on the choice of a representative ϵ) can be expressed as

$$\bar\epsilon(\mathrm{J}_1,\mathrm{J}_2)=\sum_{\dot\mu} S^{(0)}_{\mathrm{J}_1,\dot\mu}S^{(0)}_{\mathrm{J}_2,\dot\mu}(S^{(0)}_{\mathrm{J}_1\mathrm{J}_2,\dot\mu})^*\,/\,S^{(0)}_{\Omega,\dot\mu}\,. \tag{A.8}$$

Here $S^{(0)}$ denotes the matrix that governs the modular behavior of the differences $\chi_{\mu_+}-\chi_{\mu_-}$ of orbifold characters in the untwisted sector coming from the same $\mathfrak{A}$-primary. More specifically, under $\tau\mapsto-1/\tau$ these differences become linear combinations of characters $\chi_{\dot\mu}$ in the twisted sector, with the coefficients given by $S^{(0)}$ (for more details, see [16]). By the consistency of the orbifold fusion rules, $\bar\epsilon$ can only take the values ±1 and satisfies $\bar\epsilon(\mathrm{J}_1,\mathrm{J}_2)=\bar\epsilon(\mathrm{J}_2,\mathrm{J}_1)$, and therefore [72] indeed determines a unique class $[\epsilon]$ in $H^2(\mathcal{G}_0,\mathbb{Z}_2)$. In the special case where $\mathcal{G}_0=\mathcal{G}$, for the construction of $\mathfrak{A}^\omega_{\rm ext}$ we must pick, for each $\mathrm{J}\in\mathcal{G}$, one of the fields $\mathrm{J}_\pm\in\mathcal{G}^\omega$ of the ω-orbifold, in such a manner that the chosen set of representatives closes under fusion. Thus we must find a section σ: $\mathcal{G}^0\to\mathcal{G}^\omega$ for the exact sequence (A.7). Such a section exists only if the extension is trivial, i.e. if $[\epsilon]=1$. In conclusion, there is an obstruction to the lift of ω to an automorphism $\omega_{\rm ext}$ of $\mathfrak{A}_{\rm ext}$; moreover, it is controlled by the twisted sector of the orbifold, and hence computable.

For various classes of orbifold constructions the presence of an obstruction can be decided without too much effort. For instance, from the results of [16] it follows immediately that there is no obstruction when ω is an automorphism of the chiral algebra of a WZW model that comes from an inner automorphism of the underlying simple Lie algebra. Similarly, no obstruction is present for the charge conjugation orbifold of a single free boson [17] and for arbitrary permutation orbifolds. As a matter of fact, we do not know of any orbifold construction where the obstruction is present. It is tempting to expect that the obstruction is indeed absent in all cases that appear in conformal field theory, but so far we do not have any general argument to this effect. In any case, in all the applications in the main text we are able to show that the obstruction is absent. Therefore in the present paper we do not attempt to push this issue further.

The fields in $\mathcal{G}$ which are not contained in the subgroup $\mathcal{G}_0$ (A.6) come in pairs J and $\omega^*(\mathrm{J})$, and each such pair gives rise to a single field in $\mathfrak{A}^\omega$, which has quantum dimension 2, i.e. is *not* a simple current any longer. As a consequence, these fields do not have a direct influence on the presence of an obstruction. On the other hand, even when there is no obstruction, it turns out to be quite non-trivial to describe such non-simple current fields in sufficiently explicit terms in concrete models. In particular, consistency of the fusion rules of the $\mathfrak{A}$-theory does not seem to be of any help.

Acknowledgement

We are grateful to Jens Fjelstad, Jürg Fröhlich, Lennaert Huiszoon, Peter Kaste, Wolfgang Lerche, Andy Lütken, and Bert Schellekens for helpful discussions and comments.

[17] When n as introduced before formula (A.2) is odd, then $\mathcal{G}_0=\{0\}$, and the statement is trivial. When n is even, then $\mathcal{G}_0=\{0,mn^2/2\}\cong\mathbb{Z}_2$, while the simple current group $\mathcal{G}^\omega$ of the orbifold is $\mathbb{Z}_2\times\mathbb{Z}_2$, so $\mathcal{G}^\omega$ is a trivial extension of $\mathcal{G}_0$.

References

[1] D. Gepner, *Space-time supersymmetry in compactified string theory and superconformal models*, Nucl. Phys. B 296 (1988) 757

[2] D. Gepner, in: *Superstrings '89*, M. Green, R. Iengo, S. Randjbar-Daemi, E. Sezgin, and A. Strominger, eds. (World Scientific, Singapore 1990), p. 238

[3] A.N. Schellekens and S. Yankielowicz, *New modular invariants for $N=2$ tensor products and four-dimensional strings*, Nucl. Phys. B 330 (1990) 103

[4] M. Lynker and R. Schimmrigk, *A-D-E quantum Calabi–Yau manifolds*, Nucl. Phys. B 339 (1990) 121

[5] J. Fuchs, A. Klemm, C. Scheich, and M.G. Schmidt, *Spectra and symmetries of Gepner models compared to Calabi–Yau compactifications*, Ann. Phys. 204 (1990) 1

[6] R.E. Behrend, P.A. Pearce, V.B. Petkova, and J.-B. Zuber, *Boundary conditions in rational conformal field theories*, Nucl. Phys. B 570 (2000) 525

[7] M. Naka and M. Nozaki, *Boundary states in Gepner models*, preprint hep-th/0001037

[8] A.N. Schellekens and S. Yankielowicz, *Extended chiral algebras and modular invariant partition functions*, Nucl. Phys. B 327 (1989) 673

[9] K. Intriligator, *Bonus symmetry in conformal field theory*, Nucl. Phys. B 332 (1990) 541

[10] A.N. Schellekens and S. Yankielowicz, *Simple currents, modular invariants, and fixed points*, Int. J. Mod. Phys. A 5 (1990) 2903

[11] R. Dijkgraaf, C. Vafa, E. Verlinde, and H. Verlinde, *The operator algebra of orbifold models*, Commun. Math. Phys. 123 (1989) 485

[12] J. Fuchs and C. Schweigert, *Symmetry breaking boundaries II. More structures; examples*, Nucl. Phys. B 568 (2000) 543

[13] J.L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, Nucl. Phys. B 324 (1989) 581

[14] J. Fuchs and C. Schweigert, *Orbifold analysis of broken bulk symmetries*, Phys. Lett. B 447 (1999) 266

[15] J. Fuchs and C. Schweigert, *Symmetry breaking boundaries I. General theory*, Nucl. Phys. B 558 (1999) 419

[16] L. Birke, J. Fuchs, and C. Schweigert, *Symmetry breaking boundary conditions and WZW orbifolds*, Adv. Theor. Math. Phys. 3 (1999) No.3

[17] I. Brunner and V. Schomerus, *D-branes at singular curves of Calabi–Yau compactifications*, preprint hep-th/0001132

[18] F. Englert, H. Nicolai, and A.N. Schellekens, *Superstrings from 26 dimensions*, Nucl. Phys. B 274 (1986) 315

[19] A.N. Schellekens, *Multiloop modular invariance of the covariant lattice construction of fermionic strings*, Phys. Lett. B 199 (1987) 427

[20] W. Lerche, A.N. Schellekens, and N.P. Warner, *Lattices and strings*, Phys. Rep. 177 (1989) 1

[21] J. Fuchs, A.N. Schellekens, and C. Schweigert, *A matrix S for all simple current extensions*, Nucl. Phys. B 473 (1996) 323

[22] G. Felder, K. Gawędzki, and A. Kupiainen, *Spectra of Wess–Zumino–Witten models with arbitrary simple groups*, Commun. Math. Phys. 117 (1988) 127

[23] P. Bantay, *The untwisted stabilizer in simple current extensions*, Phys. Lett. B 396 (1997) 183

[24] A. Bruguières, *Catégories prémodulaires, modularisations et invariants des variétés de dimension 3*, Math. Annal. 316 (2000) 215

[25] C. Vafa, *String vacua and orbifoldized LG models*, Mod. Phys. Lett. A 4 (1989) 1169

[26] A.N. Schellekens, *Field identification fixed points in $N=2$ coset theories*, Nucl. Phys. B 366 (1991) 27

[27] W. Nahm and K. Wendland, *A hiker's guide to K3. Aspects of $N=(4,4)$ superconformal field theorywith central charge $c=6$*, preprint hep-th/9912067

[28] J. Fuchs and C. Schweigert, *Non-hermitian symmetric $N=2$ coset models, Poincaré polynomials, and string compactification*, Nucl. Phys. B 411 (1994) 181

[29] C. Dong and G. Mason, *On quantum Galois theory*, Duke Math. J. 86 (1997) 305

[30] C. Dong and G. Mason, *Quantum Galois theory for compact Lie groups*, J. Algebra 214 (1998) 92

[31] A. Hanaki, M. Miyamoto, and D. Tambara, *Quantum Galois theory for finite groups*, Duke Math. J. 97 (1999) 541

[32] G. Pradisi, A. Sagnotti, and Ya.S. Stanev, *The open descendants of non-diagonal $SU(2)$ WZW models*, Phys. Lett. B 356 (1995) 230

[33] G. Pradisi, A. Sagnotti, and Ya.S. Stanev, *Completeness conditions for boundary operators in 2D conformal field theory*, Phys. Lett. B 381 (1996) 97

[34] D.C. Lewellen, *Sewing constraints for conformal field theories on surfaces with boundaries*, Nucl. Phys. B 372 (1992) 654

[35] J. Fuchs and C. Schweigert, *Branes: from free fields to general backgrounds*, Nucl. Phys. B 530 (1998) 99

[36] M. Kato and T. Okada, *D-branes on group manifolds*, Nucl. Phys. B 499 (1997) 583

[37] A. Recknagel and V. Schomerus, *D-branes in Gepner models*, Nucl. Phys. B 531 (1998) 185

[38] H. Ooguri, Y. Oz, and Z. Yin, *D-branes on Calabi–Yau spaces and their mirrors*, Nucl. Phys. B 477 (1996) 407

[39] S. Stanciu, *D-branes in Kazama–Suzuki models*, Nucl. Phys. B 526 (1998) 295

[40] M. Gutperle and Y. Satoh, *D-branes in Gepner models and supersymmetry*, Nucl. Phys. B 543 (1999) 73

[41] J. Polchinski, *Dirichlet-branes and Ramond–Ramond charges*, Phys. Rev. Lett. 75 (1995) 4724

[42] J. Polchinski, *TASI lectures on D-branes*, preprint hep-th/9611050

[43] A. Sen, *Non-BPS states and branes in string theory*, preprint hep-th/9904207

[44] A. Lerda and R. Russo, *Stable non-BPS states in string theory: a pedagogical review*, preprint hep-th/9905006

[45] G. Felder, J. Fröhlich, J. Fuchs, and C. Schweigert, *Conformal boundary conditions and three-dimensional topological field theory*, Phys. Rev. Lett. 84 (2000) 1659

[46] G. Felder, J. Fröhlich, J. Fuchs, and C. Schweigert, *Correlation functions and boundary conditions in RCFT and three-dimensional topology*, preprint hep-th/9912239

[47] D. Gepner, *Field identification in coset conformal field theories*, Phys. Lett. B 222 (1989) 207

[48] S. Govindarajan, T. Jayaraman, and T. Sarkar, *Worldsheet approaches to D-branes on supersymmetric cycles*, preprint hep-th/9907131

[49] J.L. Cardy, *Boundary conditions in conformal field theory*, Adv. Studies in Pure Math. 19 (1989) 127

[50] J. Fuchs and C. Schweigert, *Completeness of boundary conditions for the critical three-state Potts model*, Phys. Lett. B 441 (1998) 141

[51] I. Affleck, M. Oshikawa, and H. Saleur, *Boundary critical phenomena in the three-state Potts model*, preprint cond-mat/9804117

[52] I. Brunner, M.R. Douglas, A. Lawrence, and C. Römelsberger, *D-Branes on the quintic*, preprint hep-th/9906200

[53] B.R. Greene and M.R. Plesser, *Duality in Calabi–Yau moduli space*, Nucl. Phys. B 338 (1990) 15

[54] P.S. Aspinwall and C.A. Lütken, *Geometry of mirror manifolds*, Nucl. Phys. B 353 (1991) 427

[55] R. Schimmrigk, *The construction of mirror symmetry*, preprint hep-th/9209018

[56] B.R. Greene, *String theory on Calabi–Yau manifolds*, preprint hep-th/9702155

[57] J. Fuchs and C. Schweigert, *A classifying algebra for boundary conditions*, Phys. Lett. B 414 (1997) 251

[58] L.R. Huiszoon, A.N. Schellekens, and N. Sousa, *Open descendants of non-diagonal invariants*, preprint hep-th/9911229

[59] J. Böckenhauer and D.E. Evans, *Modular invariants, graphs, and α-induction for nets of subfactors III*, Commun. Math. Phys. 205 (1999) 183

[60] J. Böckenhauer, D.E. Evans, and Y. Kawahigashi, *On α-induction, chiral generators and modular invariants for subfactors*, Commun. Math. Phys. 208 (1999) 429

[61] M. Müger, *Galois theory for braided tensor categories and the modular closure*, Adv. Math. 150 (2000) 151

[62] S. Sawin, *Jones–Witten invariants for nonsimply-connected Lie groups and the geometry of the Weyl alcove*, preprint math.QA/9905010

[63] B.R. Greene, C. Vafa, and N.P. Warner, *Calabi–Yau manifolds and renormalization group flows*, Nucl. Phys. B 324 (1989) 371

[64] E. Witten, *Phases of $N=2$ theories in two dimensions*, Nucl. Phys. B 403 (1993) 159

[65] W. Lerche, C. Vafa, and N.P. Warner, *Chiral rings in $N=2$ superconformal theories*, Nucl. Phys. B 324 (1989) 427

[66] P. Green, T. Hübsch, *Polynomial deformations and Cohomology of CY manifolds*, Commun. Math. Phys. 113 (1987) 505

[67] P.S. Aspinwall and C.A. Lütken, *Quantum Algebraic geometry of superstring compactifications*, Nucl. Phys. B 355 (1991) 482

[68] D.-E. Diaconescu and C. Römelsberger, *D-Branes and bundles on elliptic fibrations*, preprint hep-th/9910172

[69] P. Kaste, W. Lerche, C. A.Lütken, and J. Walcher, *D-branes on K3-fibrations*, preprint hep-th/9912147

[70] E. Scheidegger, *D-branes on some one- and two-parameter Calabi–Yau hypersurfaces*, preprint hep-th/9912188

[71] S. Hosono, A. Klemm, S. Theisen, and S.-T. Yau, *Mirror symmetry, mirror map and applications to Calabi–Yau hypersurfaces*, Commun. Math. Phys. 167 (1995) 301

[72] G. Karpilovsky, *Induced Modules Over Group Algebras* (North Holland Publishing Company, Amsterdam 1990)